Second-Order DC Electric Motor Model

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Nomenclature

Ω	rotation rate	v	terminal voltage
Q_m	torque	$\mathcal{R}$	resistance
P_{shaft}	shaft power	i	current
η_m	efficiency	i_o	zero-torque current
K_V	speed constant	τ	torque lag time constant
K_Q	torque constant	T	temperature of windings

1 Motor Model

1.1 Fundamental relations

The behavior of DC electric motor is described by the equivalent circuit model shown in Figure 1.

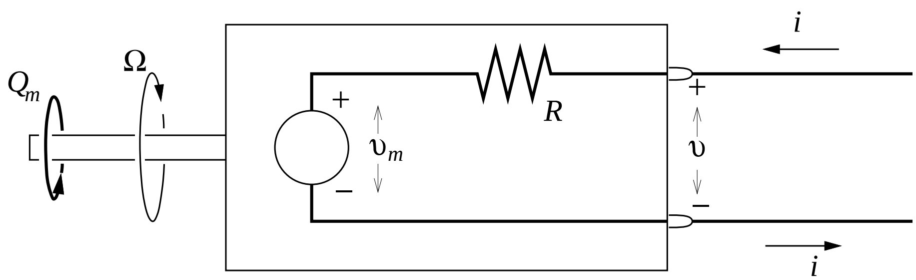

Figure 1: Equivalent circuit for a DC electric motor.

1.1.1 Resistance model

The resistance $\mathcal{R}$ depends nearly linearly on the temperature of the windings,

$$\mathcal{R} \;=\; \mathcal{R}_0(1 \,+\, \alpha\,\Delta T) \tag{1}$$

where $\mathcal{R}_0$ is the resistance at some reference temperature, ΔT is the temperature rise above the reference temperature, and α is the fractional increase in resistivity with temperature. For copper, $\alpha = 0.0042/\text{C}^\circ = 0.0023/\text{F}^\circ$. If we assume thermal equilibrium, and that ΔT is proportional to the Ohmic heating load, we can conveniently specify $\mathcal{R}$ as a function of the current,

$$\mathcal{R}(i) \;=\; \mathcal{R}_0 \,+\, \mathcal{R}_2\, i^2 \tag{2}$$

$$\mathcal{R}_2 \;=\; \mathcal{R}_0\, \alpha\, \Delta T_{\max}/i_{\max}^2 \tag{3}$$

where $\Delta T_{\max}$ is the temperature rise at the maximum rated current $i_{\max}$. Of course, assuming a constant resistance may be appropriate for many applications, in which case we simply set $\mathcal{R}_2 = 0$, so that $\mathcal{R} = \mathcal{R}_0$.

1.1.2 Torque model

The shaft torque Q_m is proportional to the current i via the torque constant K_Q, minus a friction-related current i_o.

$$Q_m(i,\Omega) \;=\; (i - i_o)/K_Q \tag{4}$$

$$i_o(\Omega) \;=\; i_{o_0} \,+\, i_{o_1}\,\Omega \,+\, i_{o_2}\,\Omega^2 \tag{5}$$

The constant coefficient i_{o_0} captures the effect of sliding friction of the brushes (if any) and bearings. The linear coefficient i_{o_1} captures laminar-flow air resistance on the rotor, and the quadratic coefficient i_{o_2} captures turbulent-flow air resistance on the rotor. Any other $i_o(\Omega)$ function could also be used without significantly complicating the overall motor model.

1.1.3 Voltage model

The internal back-EMF v_m is nearly proportional to the rotation rate Ω via the motor speed constant K_V.

$$v_m(\Omega) \;=\; (1 \,+\, \tau\Omega)\,\Omega/K_V \tag{6}$$

The additional quadratic term, scaled by the small time constant τ, represents magnetic lags in the motor. The motor terminal voltage is then obtained by adding on the resistive voltage drop.

$$v(i,\Omega) \;=\; v_{m(\Omega)} \,+\, i\,\mathcal{R}_{(i)} \tag{7}$$

1.2 Derived relations

The model equations above are now manipulated to give the current, torque, power, and efficiency, all as functions of the motor speed and terminal voltage. First, equation (7) is used to obtain the current function. For the special case of constant resistance, this is

$$i(\Omega,v) \;=\; \left[v \,-\, (1+\tau\Omega)\,\frac{\Omega}{K_V}\right]\frac{1}{\mathcal{R}} \qquad\qquad (\mathcal{R}\!=\!\text{constant}) \tag{8}$$

For the quadratic resistance function (2), equation (7) is a cubic in i, which is solvable explicitly. However, for convenience and the possibility of more general $\mathcal{R}_{(i)}$ functions, it is best to invert equation (7) by Newton's method.

$$i(\Omega,v) \;:\; v_{m(\Omega)} \,+\, i\,\mathcal{R}_{(i)} \,-\, v \;\rightarrow\; 0 \tag{9}$$

Equation (8) with $\mathcal{R} = \mathcal{R}_0$ makes a good initial guess for i to start the Newton iteration.

Regardless of how $i(\Omega,v)$ is obtained, the remaining motor variables follow immediately.

$$Q_m(\Omega,v) \;=\; \left[i_{(\Omega,v)} \;-\; i_{o(\Omega)}\right]\frac{1}{K_Q} \tag{10}$$

$$P_{\text{shaft}}(\Omega,v) \;=\; Q_m\,\Omega \tag{11}$$

$$\eta_m(\Omega,v) \;=\; \frac{P_{\text{shaft}}}{i\,v} \;=\; \left(1-\frac{i_o}{i}\right)\frac{K_V}{K_Q}\,\frac{1}{1 \,+\, \tau\Omega \,+\, i\,\mathcal{R}\,K_V/\Omega} \tag{12}$$

In the limiting case of zero friction ($i_o = 0$), zero resistive losses ($\mathcal{R} = 0$), and zero magnetic losses ($\tau = 0$), the efficiency (12) becomes

$$\eta_m = \frac{K_V}{K_Q} \qquad \text{(zero losses)} \qquad (13)$$

Hence, energy conservation requires that the torque constant K_Q must be equal to the speed constant K_V. The equations here assume K_V is in rad/s/Volt, and K_Q is in the equivalent units of Amp/Nm. However, K_V is usually given in RPM/Volt.

2 Motor Parameter Measurement

The motor operation functions (9) – (12) depend on the "motor constants" $\mathcal{R}_0$, $\mathcal{R}_2$, i_{o_0}, i_{o_1}, i_{o_2}, τ, K_V, K_Q. These can be obtained by benchtop measurements, together with simple data fitting.

2.1 Motor resistance

The motor resistance $\mathcal{R}$ can be measured directly with a milli-ohmmeter. Alternatively, it can be determined using a power supply, an ammeter, and a voltmeter. A representative range of currents i is sent through the motor by applying a suitable voltages v to the motor terminals, while the shaft is held to prevent rotation. The resistance is then computed using Ohm's Law.

$$\mathcal{R} = v/i \qquad (14)$$

On a commutated motor this will likely vary with shaft position, in which case the various $\mathcal{R}$ values need to be averaged over different shaft positions. Fitting the $\mathcal{R}$ versus i data with a parabola then determines the intercept $\mathcal{R}_0$. The quadratic coefficient of the data fit is $\mathcal{R}_2$,

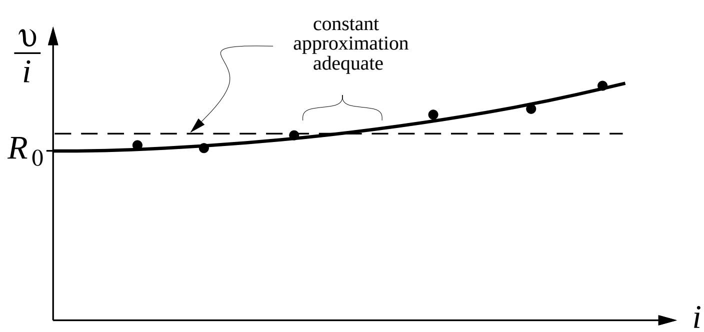

Figure 2: Resistance $\mathcal{R} = v/i$ versus stall current i, with constant or quadratic fit.

for the special case of static operation with no additional cooling. If cooling will be present, and the operating temperature is known, then it is better to estimate $\mathcal{R}_2$ from relation (3).

2.2 Zero-load current

With the motor shaft free to turn, a range of voltages v is applied to the motor, and the resulting zero-load current i_o and rotation rate Ω are measured. Fitting this $i_o(\Omega)$ data with a quadratic function gives the three motor constants i_{o_0}, i_{o_1}, i_{o_2}. By fitting only with i_{o_0} and i_{o_1}, or only with i_{o_0}, simpler linear or constant fits can also used if accuracy is required only over a narrow range of rotation rate.

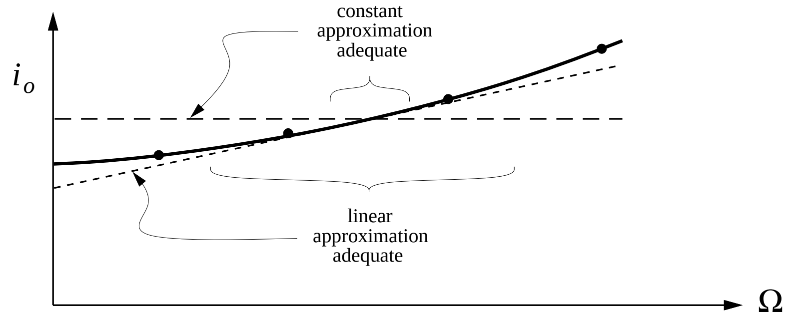

Figure 3: Zero-load current i_o data versus Ω, with constant, linear, and quadratic fits.

2.3 Speed constant and magnetic lag time constant

Using the previously-obtained resistance $\mathcal{R}$, the motor back-EMF voltage v_m can be computed for this case from the zero-load v, i_o data.

$$v_m \;=\; v - i_o \mathcal{R}(i_o) \tag{15}$$

We now consider the measured v_m/Ω ratio versus Ω. According to the (6) model, this should be a linear function of Ω.

$$v_m/\Omega \;=\; (1 + \tau\Omega)/K_V \tag{16}$$

Fitting the v_m/Ω data versus Ω then gives the intercept $1/K_V$ and the slope τ/K_V. If the data is significantly nonlinear, then some other function of Ω could be substituted for $\tau\Omega/K_V$ without significantly complicating the model.

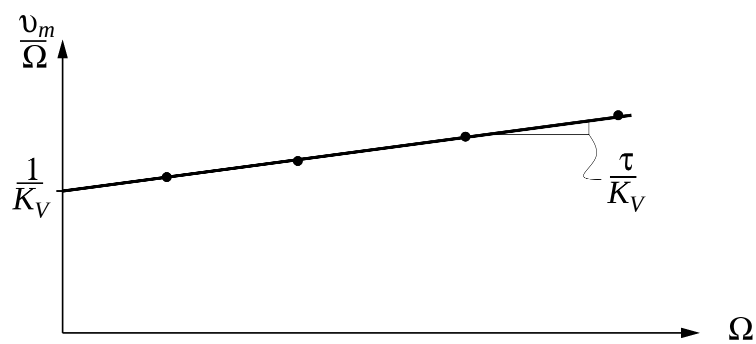

Figure 4: Back-EMF v_m/Ω data versus Ω, with curve fit to determine K_V and τ.

2.4 Torque constant

The torque constant can be simply assumed to be the same as K_V.

$$K_Q = K_V \qquad (17)$$

Alternatively, it can be obtained from motor torque data if this is available. Ideally, the motor is operated at different loads (the zero-load tests are not usable here), and over a range of speeds by applying different voltages. According to the torque model (4), the Q_m data versus $i - i_{o(\Omega)}$ should be a straight line passing through the origin. The slope of the best-fit line to the data then gives $1/K_Q$.

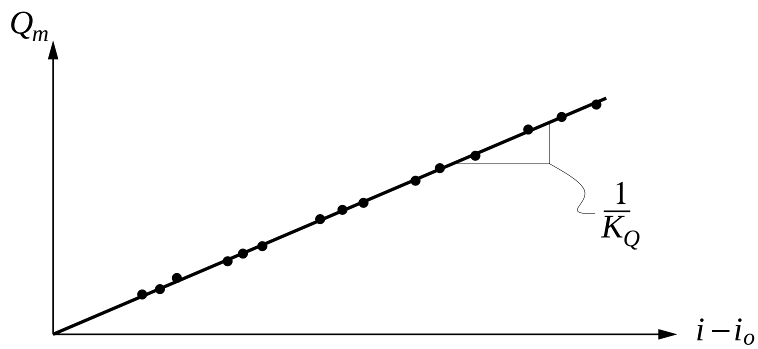

Figure 5: Q_m data versus $i - i_o$, with curve fit to determine K_Q.

As can be seen from (13), any resulting discrepancy between this measured K_Q and K_V will give a nonunity efficiency even if all the loss quantities $\mathcal{R}$, i_o, τ are set to zero. Hence, a nonunity K_V/K_Q ratio indicates the degree of imperfection of the present motor model. However, simply allowing K_Q to be different from K_V will at least partially account for the model imperfections, since the efficiency is then likely to be predicted more accurately.