QPROP Formulation

Mark Drela, MIT Aero & Astro
June 2006

This document gives the theoretical aerodynamic formulation of QPROP, which is an analysis program for predicting the performance of propeller-motor combinations. The same formulation applies to the companion design program QMIL, which generates propeller geometries for the Minimum Induced Loss (MIL) condition, or windmill geometries for the MIL or Maximum Total Power (MTP) conditions.

QPROP and QMIL use an extension of the classical blade-element/vortex formulation, developed originally by Betz [1], Goldstein [2], and Theodorsen [3], and reformulated somewhat by Larrabee [4]. The extensions include

- Radially-varying self-induction velocity which gives consistency with the heavily-loaded actuator disk limit
- Perfect consistency of the analysis and design formulations.
- Solution of the overall system by a global Newton method, which includes the self-induction effects and powerplant model.
- Formulation and implementation of the Maximum Total Power (MTP) design condition for windmills

Nomenclature

r	radial coordinate	B	number of blades
R	tip radius	V	freestream velocity
T	thrust	Ω	rotation rate
Q	torque	$u(r)$	local externally-induced velocity at disk
P	shaft power	$v(r)$	local rotor-induced velocity at disk
$\bar{\eta}$	overall efficiency	$W(r)$	local total velocity relative to blade
$\beta(r)$	local geometric blade pitch angle	$c_\ell(r)$	local blade lift coefficient
$\phi(r)$	local flow angle ($=\arctan(W_a/W_t)$)	$c_d(r)$	local blade profile drag coefficient
$\alpha(r)$	local angle of attack ($=\beta-\phi$)	λ	advance ratio ($=V/\Omega R$)
$\Gamma(r)$	local blade circulation	$\lambda_w(r)$	local wake advance ratio ($=(r/R)W_a/W_t$)
$T'(r)$	local thrust/radius	ρ	fluid density
$Q'(r)$	local torque/radius	μ	fluid viscosity
$(\,)_a$, $(\,)_t$	axial, tangential velocity components	a	fluid speed of sound

1 Flowfield velocities

1.1 Velocity decomposition

Figure 1 shows the velocity triangle seen by the blade at some radius r. The axial and tangential components of the total relative velocity W are decomposed as follows.

$$W_a = V + u_a + v_a \tag{1}$$

$$W_t = \Omega r - u_t - v_t \tag{2}$$

$$W = \sqrt{W_a^2 + W_t^2} \tag{3}$$

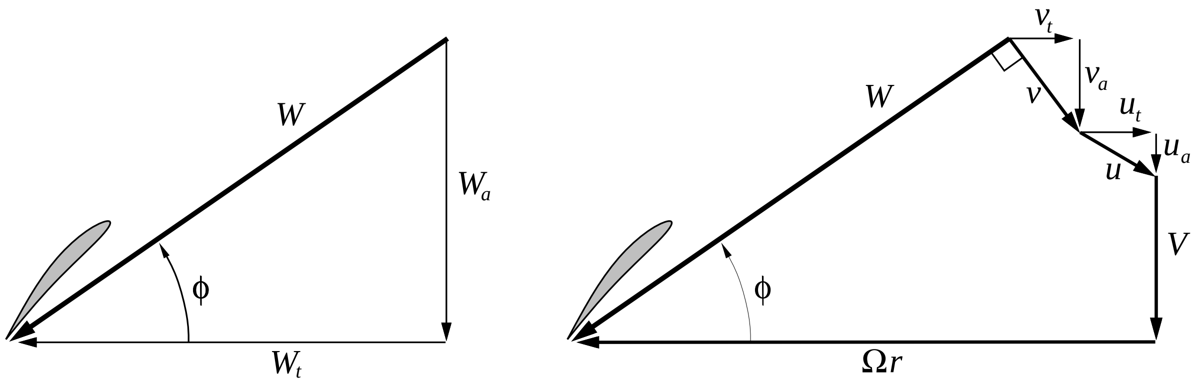

Figure 1: Decompositions of total blade-relative velocity W at radial location r.

All velocities shown in Figure 1 are taken positive as shown. Typically, v_a and v_t will be positive for a propeller with positive thrust and torque, and negative for a windmill which will have negative thrust and torque. The externally-induced u_t will be negative if it comes from an upstream counter-rotating propeller, and zero if there is no upstream torque-producing device.

1.2 Circulation/swirl relations

The tangential velocity v_t, or "swirl", is associated with the torque imparted by the rotor on the fluid. It can also be related to the circulation on the rotor blades via Helmholtz's Theorem. The total circulation on all the blades at radius r is $B\Gamma(r)$, and this must be completely shed on the blade portions inboard of r. Hence, this is also the half the circulation about a circumferential circuit in the rotor plane at radius r

$$2\pi r\,\bar{v}_t = \frac{1}{2}B\Gamma \tag{4}$$

$$\bar{v}_t = \frac{B\Gamma}{4\pi r} \tag{5}$$

where $\bar{v}_t$ is the circumferentially-averaged tangential velocity. The factor of 1/2 in equation (4) is due to the circuit seeing semi-infinite rather than infinite trailing vortices, as shown in Figure 2.

The circumferential-averaged tangential velocity $\bar{v}_t$ and the velocity on the blade v_t are assumed to be related by

$$\bar{v}_t = v_t\,F\,\sqrt{1 + (4\lambda_w R/\pi B r)^2} \tag{6}$$

The square root term becomes significant near the axis, and the modified Prandtl's factor F becomes significant near the tips and accounts for "tip losses".

$$F = \frac{2}{\pi}\arccos\left(e^{-f}\right) \tag{7}$$

$$f = \frac{B}{2}\left(1 - \frac{r}{R}\right)\frac{1}{\lambda_w} \tag{8}$$

$$\lambda_w = \frac{r}{R}\tan\phi = \frac{r}{R}\frac{W_a}{W_t} \tag{9}$$

In this modified F, the usual overall advance ratio $\lambda = V/\Omega R$ of the rotor has been replaced by the local wake advance ratio λ_w. This is more realistic for heavy disk loadings, where λ and λ_w differ considerably. The fact that λ_w varies with radius somewhat does not cause any difficulties in the formulation.

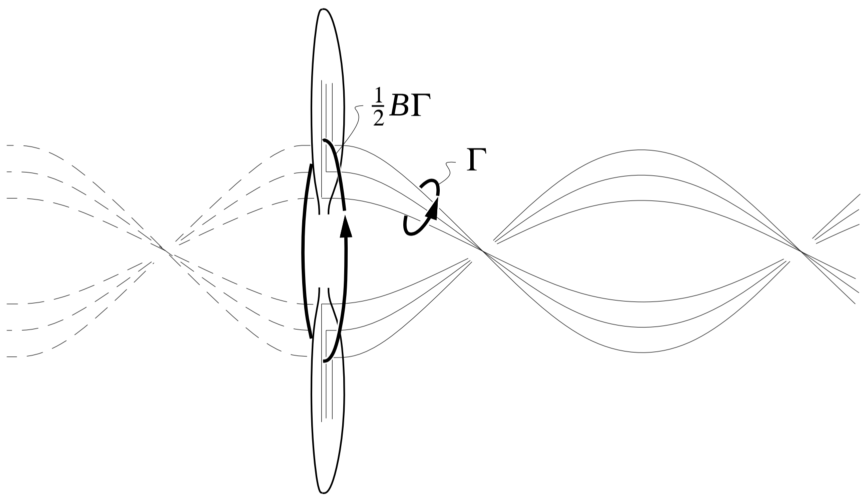

Figure 2: Circulation circuits for obtaining circulation/swirl relation.

Using the empirical relation (6) in (5) gives the final relation between the local circulation and local tangential induced velocity relative to the blade.

$$v_t = \frac{B\Gamma}{4\pi r} \frac{1}{F\sqrt{1 + (4\lambda_w R/\pi B r)^2}} \tag{10}$$

The axial induced velocity then follows with the assumption that v is perpendicular to W

$$v_a = v_t \frac{W_t}{W_a} \tag{11}$$

which is correct for the case of a non-contracting helical wake which has the same pitch at all radii. This assumption is strictly correct only for a lightly-loaded rotor having the Goldstein circulation distribution, but is expected to be reasonably good for general rotors.

It's useful to note that relations (10) and (11) are purely local. In effect, the circulation argument shown in Figure 2, together with the approximate Prandtl tip factor F, have been used in lieu of a Biot-Savart integration over the entire wake. The latter would have related the local v_t and v_a to the overall Γ distribution, and thus introduced a considerable complication.

2 Blade geometry and analysis solution

2.1 Local lift and drag coefficients

The propeller geometry and velocity triangle are shown in Figure 3.

The local angle of attack seen by the blade section is

$$\alpha(r) = \beta - \phi \tag{12}$$

$$= \beta - \arctan\frac{W_a}{W_t} \tag{13}$$

which then produces some local blade lift and profile drag coefficients.

$$c_\ell = c_\ell(\alpha, Re, Ma) \tag{14}$$

$$c_d = c_d(\alpha, Re, Ma) \tag{15}$$

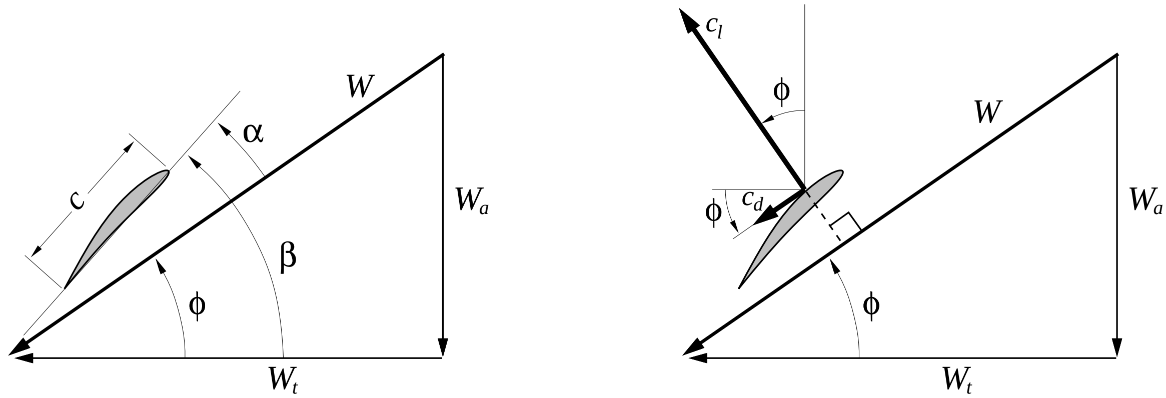

Figure 3: Blade geometry and velocity triangle at one r location.

This lift coefficient also determines the corresponding local blade circulation.

$$\Gamma \;=\; \frac{1}{2} W \, c \, c_\ell \tag{16}$$

2.2 Local analysis solution

Given some blade geometry $c(r)$, $\beta(r)$, blade airfoil properties $c_\ell(\alpha, Re, Ma)$, $c_d(\alpha, Re, Ma)$ for each radius, and operating variables V and Ω, the radial circulation distribution $\Gamma(r)$ can be calculated for each radius independently. This is performed by solving the preceding nonlinear governing equations via the Newton method. Rather than iterating on Γ directly, it is beneficial to instead iterate on the dummy variable ψ, shown in Figure 4.

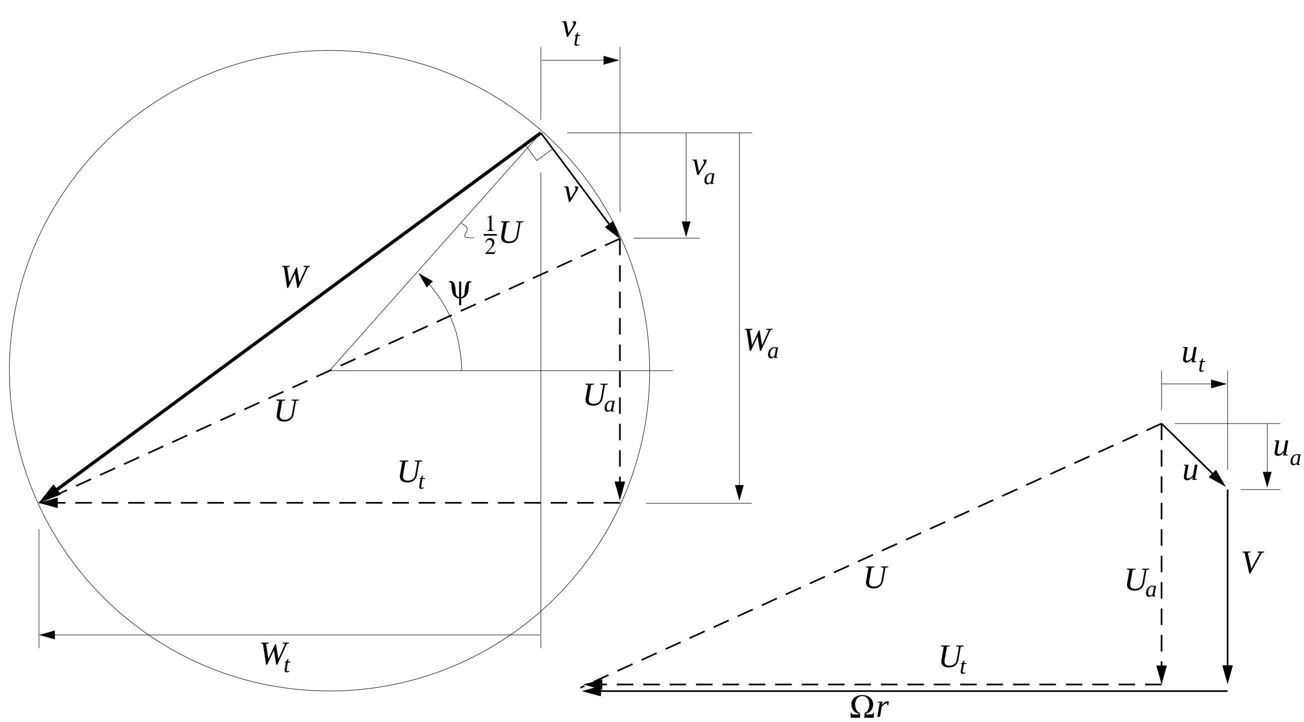

Figure 4: Velocity parameterization by the angle ψ.

This ψ parameterizes all the other variables as follows. The convenient intermediate velocity components U_a, U_t, are the overall velocities imposed on the rotor, excluding the rotor's own induced velocites v_a and v_t.

$$
\begin{aligned}
U_a &= V + u_a & (17)\\
U_t &= \Omega r - u_t & (18)\\
U &= \sqrt{U_a^2 + U_t^2} & (19)\\
W_a(\psi) &= \tfrac{1}{2}U_a + \tfrac{1}{2}U \sin\psi & (20)\\
W_t(\psi) &= \tfrac{1}{2}U_t + \tfrac{1}{2}U \cos\psi & (21)\\
v_a(\psi) &= W_a - U_a & (22)\\
v_t(\psi) &= U_t - W_t & (23)\\
\alpha(\psi) &= \beta - \arctan(W_a/W_t) & (24)\\
W(\psi) &= \sqrt{W_a^2 + W_t^2} & (25)\\
Re(\psi) &= \rho W c/\mu & (26)\\
Ma(\psi) &= W/a & (27)
\end{aligned}
$$

It is useful to note that with the above expressions, v and W are inscribed in the circle in Figure 4, and hence are always perpendicular regardless of the value of ψ.

The circulation is also related to tangential induced velocity via the Helmoltz relation (10). Again parameterizing everything with ψ, we have

$$
\begin{aligned}
\lambda_w(\psi) &= \frac{r}{R}\frac{W_a}{W_t} & (28)\\
f(\psi) &= \frac{B}{2}\left(1 - \frac{r}{R}\right)\frac{1}{\lambda_w} & (29)\\
F(\psi) &= \frac{2}{\pi}\arccos\left(e^{-f}\right) & (30)\\
\Gamma(\psi) &= v_t \frac{4\pi r}{B} F \sqrt{1 + (4\lambda_w R/\pi B r)^2} & (31)
\end{aligned}
$$

Finally, the Newton residual is the c_ℓ–Γ relation (16) recast as follows.

$$
\mathcal{R}(\psi) = \Gamma - \frac{1}{2} W c\, c_\ell(\alpha, Re, Ma) \qquad (32)
$$

A Newton update of ψ

$$
\begin{aligned}
\delta\psi &= -\frac{\mathcal{R}}{d\mathcal{R}/d\psi} & (33)\\
\psi &\leftarrow \psi + \delta\psi & (34)
\end{aligned}
$$

will then decrease $|\mathcal{R}|$ in the next Newton iteration. The convergence is quadratic, and only a few iterations are typically required to drive $\mathcal{R}$ to machine zero.

2.3 Parameter sensitivities

The simplest analysis case of the rotor has a prescribed velocity V, and also a prescribed rotation rate Ω or advance ratio λ. We might also modify all the blade angles by a constant,

$$
\beta(r) = \beta_o(r) + \Delta\beta \qquad (35)
$$

where β_o is the baseline twist distribution, and $\Delta\beta$ is the radially-constant specified pitch angle change. Each of these parameters can be set a priori, and the analysis proceeds as described above.

However, in many situations it is instead necessary to specify the overall rotor thrust or torque, or perhaps specify some torque/speed relation for a driving motor. In this case, either V, Ω or $\Delta\beta$ will be treated as an unknown. For design calculations, we wish to find the blade chord $c(r)$ and blade angle $\beta(r)$ distributions, again to achieve a specified thrust or torque. So c and β must then be treated as unknowns.

In the present approach, the analysis and design cases are treated conceptually in the same manner. The local circulation is treated as a parameter-dependent function of the form $\Gamma(r;V,\Omega,\beta,c)$, and any of its four parametric derivatives may be required for the analysis or design case.

$$\begin{aligned}
\Gamma_V(r) &\equiv \left.\frac{\partial\Gamma}{\partial V}\right|_{(\Omega,\beta,c)=\text{const}} \\
\Gamma_\Omega(r) &\equiv \left.\frac{\partial\Gamma}{\partial \Omega}\right|_{(V,\beta,c)=\text{const}} \\
\Gamma_\beta(r) &\equiv \left.\frac{\partial\Gamma}{\partial \beta}\right|_{(V,\Omega,c)=\text{const}} \\
\Gamma_c(r) &\equiv \left.\frac{\partial\Gamma}{\partial c}\right|_{(V,\Omega,\beta)=\text{const}}
\end{aligned}$$

We first note that the residual $\mathcal{R}$ defined by (32) is a function of not just ψ, but also the four parameters being considered. A physical solution requires that $\mathcal{R}$ remain at zero for all cases,

$$\mathcal{R}(\psi;V,\Omega,\beta,c) \;=\; 0 \tag{36}$$

which implicitly defines the $\psi(V,\Omega,\beta,c)$ function. The actual value of ψ is obtained numerically from (36) by the Newton iteration procedure described previously. Its parametric derivatives are then derived by setting the variation of $\mathcal{R}$ to zero, since (36) must hold for any physical perturbation.

$$\delta\mathcal{R} \;=\; \frac{\partial\mathcal{R}}{\partial\psi}\,\delta\psi \;+\; \frac{\partial\mathcal{R}}{\partial V}\,\delta V \;+\; \frac{\partial\mathcal{R}}{\partial\Omega}\,\delta\Omega \;+\; \frac{\partial\mathcal{R}}{\partial\beta}\,\delta\beta \;+\; \frac{\partial\mathcal{R}}{\partial c}\,\delta c \;=\; 0 \tag{37}$$

$$\delta\psi \;=\; -\frac{\partial\mathcal{R}/\partial V}{\partial\mathcal{R}/\partial\psi}\,\delta V \;-\; \frac{\partial\mathcal{R}/\partial\Omega}{\partial\mathcal{R}/\partial\psi}\,\delta\Omega \;-\; \frac{\partial\mathcal{R}/\partial\beta}{\partial\mathcal{R}/\partial\psi}\,\delta\beta \;-\; \frac{\partial\mathcal{R}/\partial c}{\partial\mathcal{R}/\partial\psi}\,\delta c \tag{38}$$

The variation coefficients on the righthand side of (38) are seen to be the derivatives of $\psi(V,\Omega,\beta,c)$.

$$\frac{\partial\psi}{\partial V} \;=\; -\frac{\partial\mathcal{R}/\partial V}{\partial\mathcal{R}/\partial\psi} \tag{39}$$

$$\frac{\partial\psi}{\partial \Omega} \;=\; -\frac{\partial\mathcal{R}/\partial \Omega}{\partial\mathcal{R}/\partial\psi} \tag{40}$$

$$\frac{\partial\psi}{\partial \beta} \;=\; -\frac{\partial\mathcal{R}/\partial \beta}{\partial\mathcal{R}/\partial\psi} \tag{41}$$

$$\frac{\partial\psi}{\partial c} \;=\; -\frac{\partial\mathcal{R}/\partial c}{\partial\mathcal{R}/\partial\psi} \tag{42}$$

With these, the parametric derivatives of any converged quantity can now be computed via the chain rule. For example, the circulation and its parametric derivatives are computed as follows.

$$\Gamma(\psi;V,\Omega,\beta,c) \;=\; v_t\,\frac{4}{B}\,F\,\sqrt{(\pi r)^2 \;+\; (4\lambda_w R/B)^2} \tag{43}$$

$$\Gamma_V = \frac{\partial \Gamma}{\partial V} + \frac{\partial \Gamma}{\partial \psi}\frac{\partial \psi}{\partial V} \tag{44}$$

$$\Gamma_\Omega = \frac{\partial \Gamma}{\partial \Omega} + \frac{\partial \Gamma}{\partial \psi}\frac{\partial \psi}{\partial \Omega} \tag{45}$$

$$\Gamma_\beta = \frac{\partial \Gamma}{\partial \beta} + \frac{\partial \Gamma}{\partial \psi}\frac{\partial \psi}{\partial \beta} \tag{46}$$

$$\Gamma_c = \frac{\partial \Gamma}{\partial c} + \frac{\partial \Gamma}{\partial \psi}\frac{\partial \psi}{\partial c} \tag{47}$$

Parametric derivatives of other required quantities

$$W_{a(V,\Omega,\beta,c)} \quad , \quad W_{t(V,\Omega,\beta,c)} \quad , \quad c_{\ell(V,\Omega,\beta,c)} \quad , \quad c_{d(V,\Omega,\beta,c)}$$

are obtained by this same approach. Although the blade angle variables are defined to be $\beta_{o(r)}$ and $\Delta\beta$, only the one β-derivative is needed for both, since from (35) we see that $\partial/\partial\beta = \partial/\partial\beta_o = \partial/\partial\Delta\beta$.

3 Thrust and torque relations

After the Newton iteration procedure described previously is performed for each radial station, the overall circulation distribution $\Gamma(r)$ is known. This then allows calculation of the overall thrust and torque of the rotor as follows.

3.1 Local loading and efficiencies

As shown in Figure 5, the blade lift and drag forces are resolved into thrust and torque components by using the net flow angle ϕ.

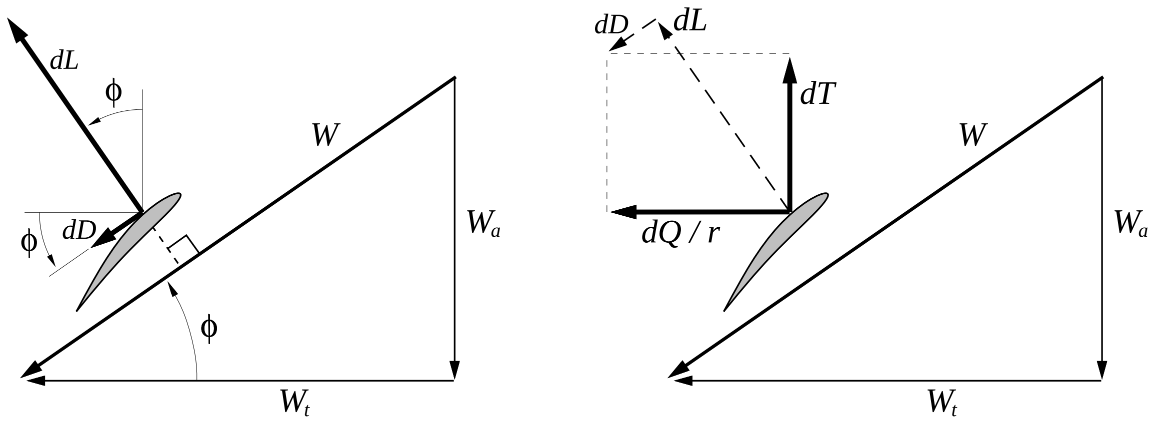

Figure 5: Blade lift and drag resolved into thrust and torque components.

$$dL = B\,\frac{1}{2}\rho W^2\, c_\ell\, c\, dr \tag{48}$$

$$dD = B\,\frac{1}{2}\rho W^2\, c_d\, c\, dr \tag{49}$$

$$\begin{aligned} dT &= dL\,\cos\phi \,-\, dD\,\sin\phi \\ &= B\,\frac{1}{2}\rho W^2\,(c_\ell\,\cos\phi \,-\, c_d\,\sin\phi)\; c\, dr \end{aligned} \tag{50}$$

$$\begin{aligned} dQ &= (dL \sin\phi \,+\, dD \cos\phi)\, r \\ &= B\,\frac{1}{2}\rho W^2 \,(c_\ell \sin\phi \,+\, c_d \cos\phi)\; c\, r\, dr \end{aligned} \tag{51}$$

From the velocity triangle in Figure 5 we see that

$$W \cos\phi = W_t \tag{52}$$

$$W \sin\phi = W_a \tag{53}$$

so the thrust and torque components can be alternatively expressed in terms of the circulation and net velocity components.

$$dT = \rho B \Gamma \left(W_t - \epsilon W_a\right)\, dr \tag{54}$$

$$dQ = \rho B \Gamma \left(W_a + \epsilon W_t\right)\, r\, dr \tag{55}$$

$$\text{where} \qquad \epsilon = \frac{c_d}{c_\ell} \tag{56}$$

The local efficiency is

$$\eta = \frac{V\, dT}{\Omega\, dQ} \tag{57}$$

$$= \frac{V}{\Omega r}\, \frac{c_\ell \cos\phi \,-\, c_d \sin\phi}{c_\ell \sin\phi \,+\, c_d \cos\phi} \tag{58}$$

$$= \frac{V}{\Omega r}\, \frac{W_t - \epsilon W_a}{W_a + \epsilon W_t} \tag{59}$$

which can be decomposed into induced and profile efficiencies.

$$\eta = \eta_i\, \eta_p \tag{60}$$

$$\eta_i = \frac{1 - v_t/U_t}{1 + v_a/U_a} \tag{61}$$

$$\eta_p = \frac{1 - \epsilon W_a/W_t}{1 + \epsilon W_t/W_a} \tag{62}$$

In the limits $V/\Omega R \!\rightarrow\! 0$, $\epsilon \!\rightarrow\! 0$, and with $u_a \!=\! u_t \!=\! 0$ (no externally-induced velocity), the efficiency reduces to

$$\eta \;\rightarrow\; \frac{1}{1 + v_a/V} \tag{63}$$

which exactly corresponds to the actuator disk limit, even for arbitrarily large disk loadings.

3.2 Total loads and efficiency

The total thrust and torque are obtained by integrating (54) and (55) along the blade.

$$T = \rho B \int_0^R \Gamma \left(W_t - \epsilon W_a\right)\, dr \;\simeq\; \rho B \sum_r \Gamma \left(W_t - \epsilon W_a\right)\, \Delta r \tag{64}$$

$$Q = \rho B \int_0^R \Gamma \left(W_a + \epsilon W_t\right)\, r\, dr \;\simeq\; \rho B \sum_r \Gamma \left(W_a + \epsilon W_t\right)\, r\, \Delta r \tag{65}$$

The simple midpoint rule is used for the integral summations. The overall efficiency is then

$$\bar{\eta} = \frac{V\, T}{\Omega\, Q} \tag{66}$$

3.3 Parametric derivatives

In order to drive an analysis solution to a specified thrust or torque, their values defined by (64) and (65) must be considered to be functions of the form $T(V,\Omega,\Delta\beta)$, $Q(V,\Omega,\Delta\beta)$. Their derivatives, which are required for Newton iteration, are obtained by implicit differentiation inside the summations, and by using the previously derived parametric derivatives of Γ, W_a, and W_t. For example, for the V-derivatives of T and Q we have

$$T_V = \rho B \sum_r \left[\Gamma_V \left(W_t - \epsilon W_a\right) + \Gamma \left(W_{t_V} - \epsilon W_{a_V} - \epsilon_V W_a\right)\right] \Delta r \tag{67}$$

$$Q_V = \rho B \sum_r \left[\Gamma_V \left(W_a + \epsilon W_t\right) + \Gamma \left(W_{a_V} + \epsilon W_{t_V} + \epsilon_V W_t\right)\right] r \, \Delta r \tag{68}$$

$$\text{where} \quad \epsilon_V = \frac{c_{d_V}}{c_\ell} - \epsilon \frac{c_{\ell_V}}{c_\ell} \tag{69}$$

The Ω- and $\Delta\beta$-derivatives of T and Q are computed in the same manner.

In order to drive a design solution to a specified thrust or torque, their values defined by (64) and (65) must be considered to be functions of the form $T(\beta,c)$, $Q(\beta,c)$. Their β- and c-derivatives, which are defined for each radial station, are obtained simply by taking the one summation term for that radius, e.g.

$$T_\beta = \rho B \left[\Gamma_\beta \left(W_t - \epsilon W_a\right) + \Gamma \left(W_{t_\beta} - \epsilon W_{a_\beta} - \epsilon_\beta W_a\right)\right] \Delta r \tag{70}$$

and likewise for Q_β, T_c, Q_c.

4 Analysis

The analysis problem is to determine the loading on a rotor of given geometry and airfoil properties, with some suitable imposed operating conditions. The unknowns are taken to be $\Gamma(r)$, V, Ω, $\Delta\beta$. The constraints on $\Gamma(r)$ are always the Newton residuals defined previously by (32) at each radial station.

$$\mathcal{R}_{r}(\Gamma(r)) = \Gamma - \frac{1}{2} W \, c \, c_\ell \tag{32}$$

The three residuals which constrain the remaining three variables V, Ω, $\Delta\beta$ will depend on the type of analysis problem being solved. Some typical constraint combinations might be:

1) Specify velocity, RPM, pitch:

$$\begin{aligned} \mathcal{R}_1(V,\Omega,\Delta\beta) &= V - V_{\text{spec}} \\ \mathcal{R}_2(V,\Omega,\Delta\beta) &= \Omega - \Omega_{\text{spec}} \\ \mathcal{R}_3(V,\Omega,\Delta\beta) &= \Delta\beta - \Delta\beta_{\text{spec}} \end{aligned} \tag{71}$$

2) Specify velocity, RPM, torque (find pitch of constant-speed prop to match engine torque):

$$\begin{aligned} \mathcal{R}_1(V,\Omega,\Delta\beta) &= V - V_{\text{spec}} \\ \mathcal{R}_2(V,\Omega,\Delta\beta) &= \Omega - \Omega_{\text{spec}} \\ \mathcal{R}_3(V,\Omega,\Delta\beta) &= Q - Q_{\text{spec}} \end{aligned} \tag{72}$$

3) Specify velocity, pitch, thrust (find RPM to get a required thrust):

$$\begin{aligned} \mathcal{R}_1(V,\Omega,\Delta\beta) &= V - V_{\rm spec} \\ \mathcal{R}_2(V,\Omega,\Delta\beta) &= \Delta\beta - \Delta\beta_{\rm spec} \\ \mathcal{R}_3(V,\Omega,\Delta\beta) &= T - T_{\rm spec} \end{aligned} \tag{73}$$

4) Specify RPM, pitch, thrust (find velocity where required thrust occurs):

$$\begin{aligned} \mathcal{R}_1(V,\Omega,\Delta\beta) &= \Omega - \Omega_{\rm spec} \\ \mathcal{R}_2(V,\Omega,\Delta\beta) &= \Delta\beta - \Delta\beta_{\rm spec} \\ \mathcal{R}_3(V,\Omega,\Delta\beta) &= T - T_{\rm spec} \end{aligned} \tag{74}$$

The three chosen residuals are simultaneously driven to zero by multivariable Newton iteration.

$$\left\{ \begin{array}{c} \delta V \\ \delta\Omega \\ \delta\Delta\beta \end{array} \right\} = - \left[\frac{\partial(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3)}{\partial(V,\Omega,\Delta\beta)} \right]^{-1} \left\{ \begin{array}{c} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \end{array} \right\} \tag{75}$$

If $\mathcal{R}_3$ is either a T or Q constraint, its Jacobian matrix entries are already known from the parametric derivatives of $T(V,\Omega,\Delta\beta)$ or $Q(V,\Omega,\Delta\beta)$, computed as described previously.

5 Design

The design problem is to determine the geometry of a rotor which matches specified parameters, typically R, V, Ω, and either T or Q. The blade airfoil properties are also specified. The two unknowns at each radial station are taken to be $\beta(r)$ and $c(r)$, which therefore requires two constraints to be imposed at each radial station. One such constraint, used for all design cases, is simply a specified local lift coefficient.

$$\mathcal{R}_{r_1} = c_\ell - c_{\ell_{\rm spec}} \tag{76}$$

For the second constraint, different options are used depending on the type of design problem being solved.

5.1 Minimum Induced Loss

In this design case, the local induced efficiency is required to be radially constant, and equal to some initially-unknown value $\bar{\eta}$. Setting $\epsilon = 0$ in equation (59) gives a suitable expression for the induced efficiency for this constraint.

$$\frac{V}{\Omega r}\frac{W_t}{W_a} = \bar{\eta} \tag{77}$$

$$\mathcal{R}_{r_2} = \bar{\eta}\, \Omega r\, W_a - V\, W_t \tag{78}$$

5.2 Overall load constraint

The design residual (78) has introduced $\tilde{\eta}$ as one additional unknown. This in effect controls the overall propeller load, since a large thrust or torque typically implies a small induced efficiency, and vice versa. Therefore, a suitable constraint for $\tilde{\eta}$ is to impose a specified thrust or torque.

$$\mathcal{R}_\eta = T - T_{\rm spec} \tag{79}$$

$$\text{or} \quad \mathcal{R}_\eta = Q - Q_{\rm spec} \tag{80}$$

Since Ω is assumed to be given, specifying Q is equivalent to specifying the shaft power $P = \Omega Q$.

Imposing $\mathcal{R}_{r_1} = 0$ and $\mathcal{R}_{r_2} = 0$ at each radial location, together with the one additional $\mathcal{R}_\eta = 0$, will give a propeller with a *Minimum Induced Loss* (MIL). Such a propeller will have the maximum possible overall induced efficiency for its radius and operating conditions. It is the analog of the elliptically-loaded wing, which has a spanwise-constant induced-drag/lift ratio.

5.3 Maximum windmill power

The MIL design residual (78) is applicable to a windmill, provided $c_\ell(r)$, and T or Q are set negative. This will give a windmill with the most shaft power for a given tower load, or the minimum tower load for a given shaft power.

An alternative design objective for a windmill is to simply maximize shaft power, regardless of the tower load. Actuator disk theory predicts that in the limits $\lambda \to 0$, $\epsilon \to 0$, the maximum (most negative) shaft power is

$$P_{\max} = -\frac{8}{27} \rho V^3 \pi R^2$$

In the presence of profile and swirl losses, the actual power will be less than this. It is of great interest to maximize the power in the presence of these losses.

With a specified Ω, maximizing P is equivalent to maximizing the torque Q. Because all the radial blade stations are assumed independent, it is sufficient to maximize the differential torque dQ. From equation (55), and the Γ–v_t relation (10) we have the following.

$$dQ = \rho B \Gamma \left(W_a + \epsilon W_t \right) \, r \, dr \tag{81}$$

$$dQ = \rho \, v_t \, \left(W_a + \epsilon W_t \right) \, 4\pi \, r^2 \, F \sqrt{1 + (4\lambda_w R / \pi B r)^2} \, dr \tag{82}$$

Figure 6 illustrates how this dQ varies for the inviscid case, with v_t negative as for a windmill. The product $dQ \sim v_t W_a$ clearly has an extremum at an intermediate loading. Since all the quantities in the dQ expression above are parameterized by ψ, we can extremize dQ by setting its ψ–derivative to zero, taking the log first for algebraic simplicity.

$$\ln(dQ) = \ln v_t + \ln\left(W_a + \epsilon W_t \right) + \ln F + \ln(4\pi \, r^2 \rho \sqrt{1 + (4\lambda_w R / \pi B r)^2} \, dr) \tag{83}$$

$$\frac{dQ'}{dQ} = \frac{v_t'}{v_t} + \frac{W_a' + \epsilon W_t' + \epsilon' W_t}{W_a + \epsilon W_t} + \frac{F'}{F} = 0 \tag{84}$$

$$\text{where} \qquad (\,)' \equiv \frac{\partial(\,)}{\partial \psi}$$

In practice, both F and ϵ are very nearly independent of ψ, so that $F' \simeq 0$ and $\epsilon' \simeq 0$. Also, from (20), (21), (23) we have

$$W_a' = \tfrac{1}{2} U \cos\psi = W_t - \tfrac{1}{2} U_t \tag{85}$$

$$W_t' = -\tfrac{1}{2} U \sin\psi = \tfrac{1}{2} U_a - W_a \tag{86}$$

$$v_t' = \tfrac{1}{2} U \sin\psi = W_a - \tfrac{1}{2} U_a \tag{87}$$

so that (84) can be simplified to the following residual.

$$\mathcal{R}_{r_2} = \frac{W_a - \frac{1}{2} U_a}{U_t - W_t} + \frac{W_t - \frac{1}{2} U_t - \epsilon (W_a - \frac{1}{2} U_a)}{W_a + \epsilon W_t} \tag{88}$$

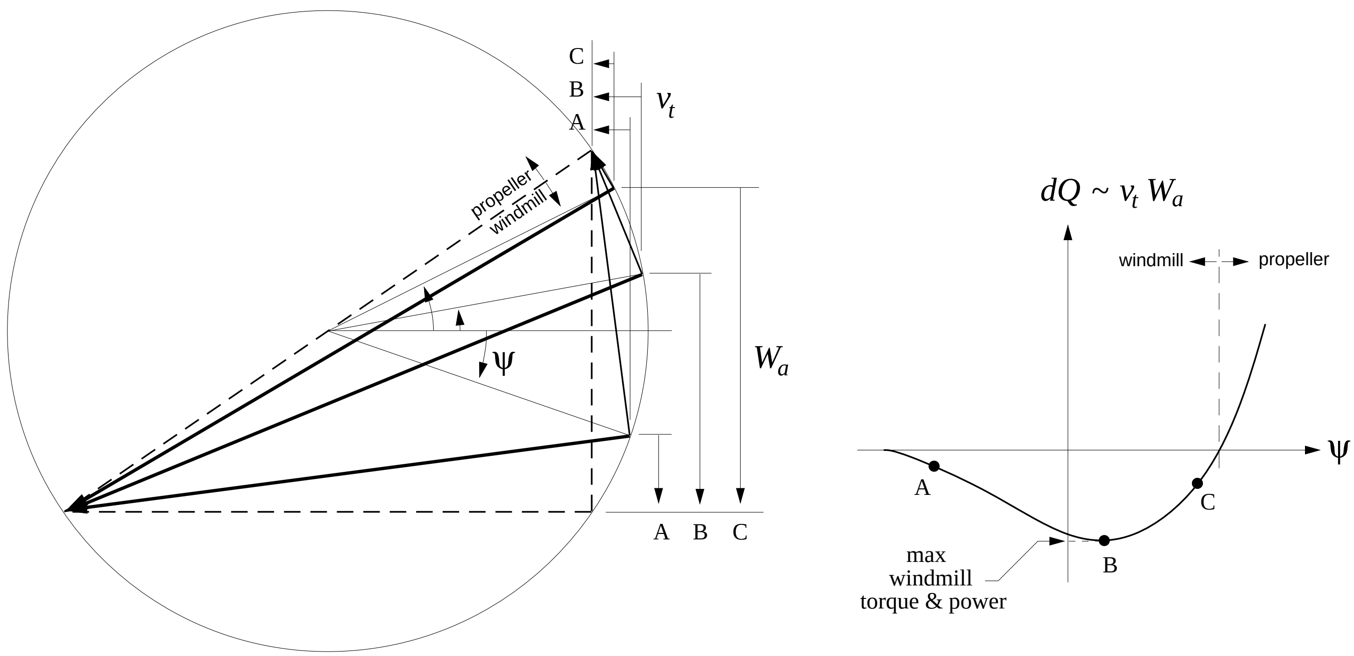

Figure 6: Three different windmill loadings. The windmill torque and power are maximum at loading B.

By taking the usual blade geometry variables $\beta(r)$ and $c(r)$ as unknowns, and driving the residuals (76) and (88) to zero at each radial location, produces a windmill with a *Maximum Total Power* (MTP). Both T and Q are a result of this calculation, and a total load constraint such as (79) or (80) in the MIL case is not required here.

5.4 Moderated maximum windmill power

We now consider designing a windmill which intentionally delivers less power than the theoretical maximum at point B in Figure 6. Such a "moderated" optimum is point B' in Figure 7.

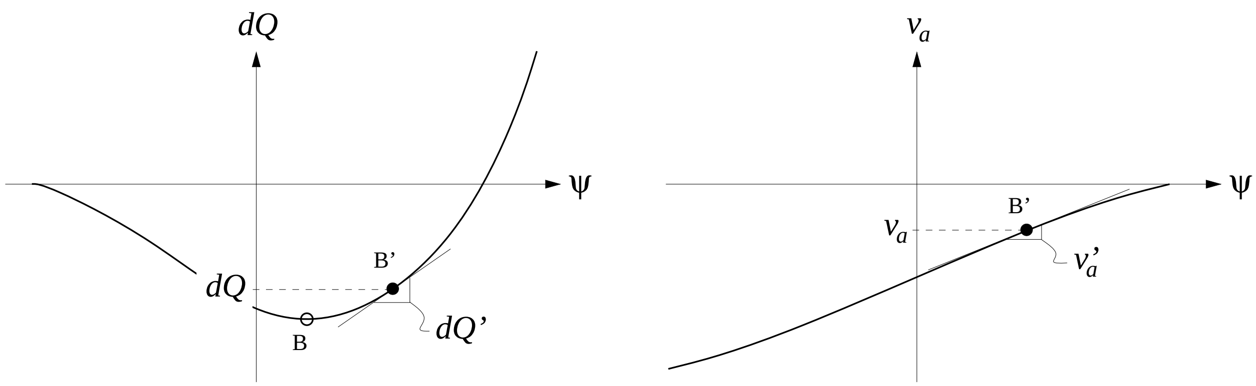

Figure 7: Moderated sub-optimum windmill power at point B'. Axial velocity gradient v_a' is used to normalize nonzero dQ'.

This moderated optimum is selected by specifying some nonzero value for the local fractional torque gradient dQ'/dQ. Because a suitable value is not easily determined a priori, it is implemented to be a specified multiple of a reference gradient. A convenient choice for this reference is v_a'/v_a, since v_a' varies little over a significant range of ψ values, as can be seen in Figure 7. This reference ratio is readily evaluated from equation (20) and (22).

$$v_a = W_a - U_a = -\tfrac{1}{2}U_a + \tfrac{1}{2}U \sin\psi \qquad (89)$$

$$v_a' = \tfrac{1}{2}U \cos\psi = W_t - \tfrac{1}{2}U_t \qquad (90)$$

The maximum-power statement (84) is now replaced by

$$\frac{dQ'/dQ}{v_a'/v_a} = K \qquad (91)$$

where K is a constant which specifies the degree of deviation away from the optimum. With substitutions for dQ'/dQ and v_a'/v_a, equation (91) is recast as an alternative residual which replaces (88).

$$\mathcal{R}_{r_2} = \left[\frac{W_a - \frac{1}{2}U_a}{U_t - W_t} + \frac{W_t - \frac{1}{2}U_t - \epsilon(W_a - \frac{1}{2}U_a)}{W_a + \epsilon W_t}\right] \frac{W_a - U_a}{W_t - \frac{1}{2}U_t} - K \qquad (92)$$

The constant $K = \mathcal{O}(1)$ is set by the designer. Choosing $K = 0$ recovers the true maximum power case, while choosing $K > 0$ gives progressively smaller power values.

The advantage of a moderated optimum stems from practical reasons. Because the torque Q and the power $P = Q\Omega$ are stationary with respect to K, the loss in available power scales as

$$P - P_0 \sim K^2$$

where P_0 is the maximum power at $K = 0$. In contrast, the thrust T (or tower load) and typical blade chord c differences change linearly with K.

$$T - T_0 \sim K$$

$$c - c_0 \sim K$$

Figure 8 shows these variations quantitatively for a typical windmill. Choosing $K = 0.2$, for instance, will incur only a 2.3% power penalty, but will give 8.5% smaller tower load and blade chords. The smaller tower load is likely to give the windmill a higher maximum safe operating windspeed, so that the moderated-optimum windmill may actually produce more time-averaged power if maximum safe winds are occasionally encountered. Alternatively, for a given tower load or windmill material cost, the moderated-optimum windmill can have a slightly larger diameter and hence actually produce more power. It is therefore certain that the moderated optimum may be better than the true fixed-diameter and fixed-windspeed optimum, when overall system costs and other operating considerations are taken into account.

References

[1] A. Betz. Airscrews with minimum energy loss. Report, Kaiser Wilhelm Institute for Flow Research, 1919.

[2] S. Goldstein. On the vortex theory of screw propellers. *Proceedings of the Royal Society*, 123, 1929.

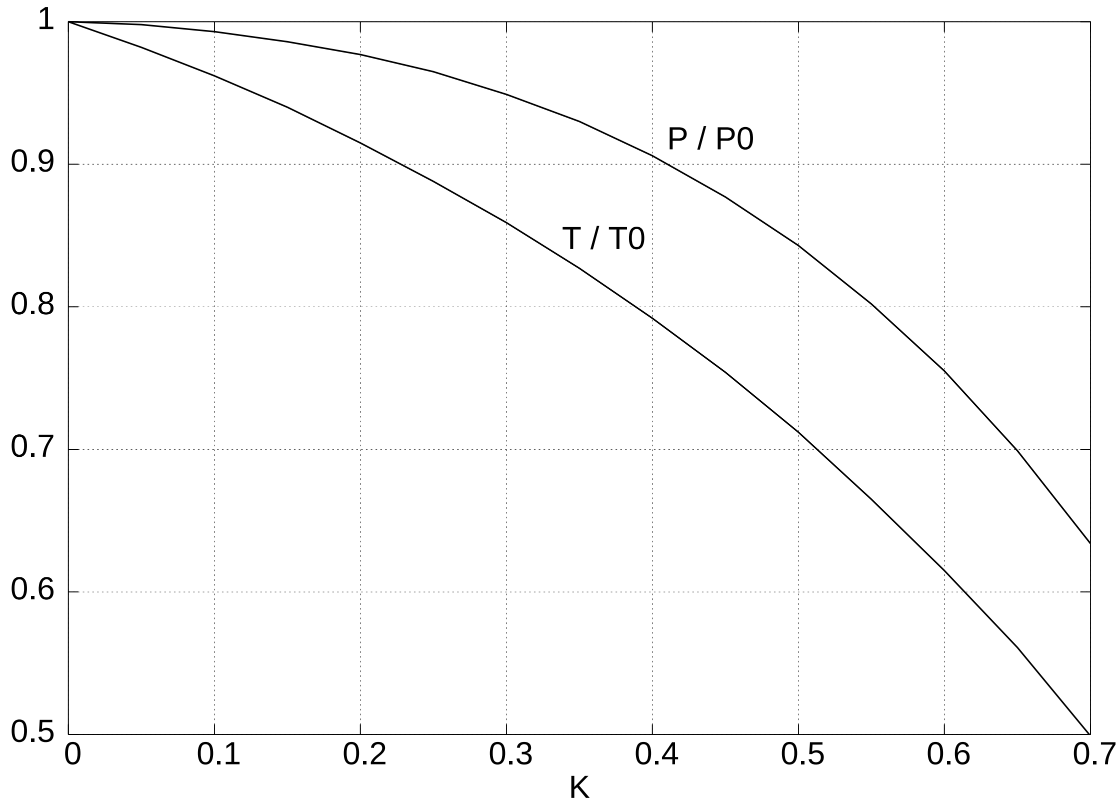

Figure 8: Variation of power P and thrust T with power-moderating constant K, for four-bladed windmill with $V/\Omega R = 0.125$. Blade chord ratio c/c_0 is nearly equal to T/T_0.

[3] T. Theodorsen. *Theory of Propellers*. McGraw-Hill, New York, 1948.

[4] E.E. Larrabee and S.E. French. Minimum induced loss windmills and propellers. *Journal of Wind Engineering and Industrial Aerodynamics*, 15:317–327, 1983.