18.01a Exam 1, ESG Fall 2007 Solutions

Problem 1. (10 points)

$(1+x)^{1/3} \approx 1 + \frac{1}{3}x - \frac{1}{9}x^2$ for $x \approx 0 \;\Rightarrow\; (1.1)^{1/3} \approx \boxed{1 + \frac{1}{30} - \frac{1}{900}.}$

Problem 2. (10 points)

A.) (7) Taylor series for $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\Rightarrow \boxed{\cos\sqrt{x} \approx 1 - \frac{x}{2!} + \frac{x^2}{4!}.}$

B.) (3) Error $\approx$ next non-zero term in Taylor series $= \frac{x^3}{6!}$.

$x = 10^{-4} \;\Rightarrow\;$ error $\approx \frac{10^{-12}}{720} \approx 10^{-15}$: $\boxed{\textbf{answer: } n = 15.}$

Problem 3. (15 points)

A.) (8) We know: $\lim_{x\to 0} \frac{\cos x}{x^2+1} = 1$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1 \;\Rightarrow\; \lim(\text{product}) = \boxed{1}$.

B.) (7) $L = \lim_{x\to\infty} x^2 e^{-x} = \infty \cdot 0.$

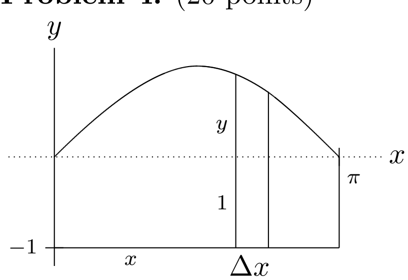
Rewrite: $L = \lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}.$

$\Rightarrow$ use L'Hospital: $L = \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty}.$

$\Rightarrow$ L'Hospital again: $L = \lim_{x\to\infty} \frac{2}{e^x} = 0.$

answer: $\boxed{L = 0.}$

Problem 4. (20 points)



Make a slice:

x = position of slice on x-axis.

$h = \sin x + 1$ = height of slice (from $y = -1$ to graph).

Δx = width of slice.

A.) (5) $dA = (y+1)\,dx \;\Rightarrow\; A = \int_0^\pi \sin(x) + 1\,dx.$

B.) (5) Shells: $dV = 2\pi x h\,dx \;\Rightarrow\; V = \int_0^\pi 2\pi x(\sin x + 1)\,dx.$

C.) (5) Disks: $dV = \pi h^2\,dx \;\Rightarrow\; V = \int_0^\pi \pi(\sin x + 1)^2\,dx.$

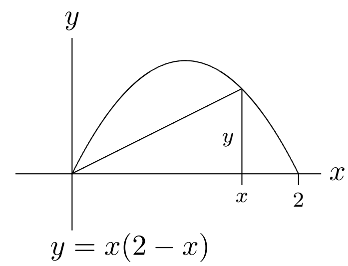
D.) (5) $ds = \sqrt{1 + (y')^2}\,dx = \sqrt{1 + \cos^2 x}\,dx \;\Rightarrow\; L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx.$

(continued)

Problem 5. (10 points) Use a u substitution: $u = \cos x \Rightarrow du = -\sin x\,dx$
Changing limits: $x = 0 \Rightarrow u = 1, \quad x = \pi/4 \Rightarrow u = 1/\sqrt{2}$

$$\Rightarrow \int_0^{\pi/4} \frac{\sin x}{\cos^7 x}\,dx = \int_1^{1/\sqrt{2}} \frac{-du}{u^7}\,du = \int_{\frac{1}{\sqrt{2}}}^1 \frac{du}{u^7} = -\frac{1}{6}u^{-6}\Big|_{1/\sqrt{2}}^1 = -\frac{1}{6}(1-8) = \boxed{\frac{7}{6}.}$$

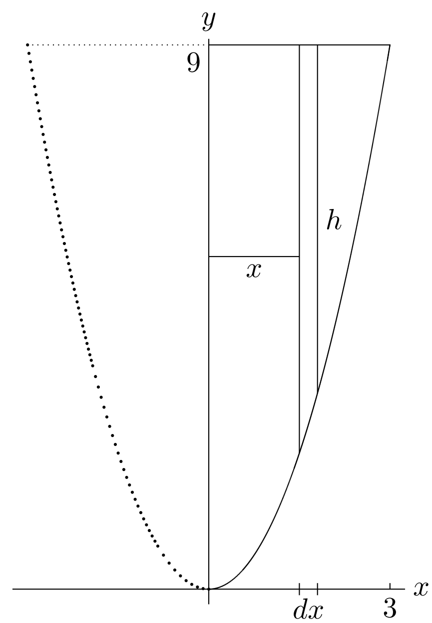
Problem 6. (10 points)



Area $= \frac{1}{2}xy$
We want to average with respect to x.

$$\Rightarrow \text{ Average area } = \frac{1}{2}\int_0^2 \frac{1}{2}xy\,dx = \boxed{\frac{1}{4}\int_0^2 x\,x(2-x),dx.}$$

Problem 7. (10 points)
We slice by shells.
Note each shell has a constant density (i.e. it's at a fixed distance from the y-axis.



Using the picture, we get:
Volume of shell $= dV = 2\pi x h\,dx = 2\pi x(9-x^2)\,dx$.
Density of shell $= \delta(x) = \sqrt{x}$.
$\Rightarrow$ mass of shell $= dm = \delta\,dV = 2\pi\,(\sqrt{x})\,x(9-x^2)\,dx$.

$$\begin{aligned}\Rightarrow \text{ total mass } &= \int_0^3 2\pi\,(\sqrt{x})\,x(9-x^2)\,dx \\ &= 2\pi\int_0^3 9x^{3/2} - x^{7/2}\,dx \\ &= 2\pi\left(\frac{18}{5}x^{5/2} - \frac{2}{9}x^{9/2}\Big|_0^3\right. \\ &= \boxed{2\pi\left(\frac{18}{5}\cdot 3^{5/2} - \frac{2}{9}\cdot 3^{9/2}\right).}\end{aligned}$$

Problem 8. (15 points)
A.) (5) $\ln 1 = 0 \Rightarrow F(x)$ is defined and differentiable for $x > 1$.

B.) (5) The limit is indeterminant of form ∞/∞.

L'Hospital $\Rightarrow$ limit $= \lim_{x\to\infty} \frac{1/x}{1/\ln x} = \lim_{x\to\infty}\frac{\ln x}{x} = 0.$

We know the last limit is 0 because $x >> \ln x$. (Or we could use L'Hospital.)

C.) (5) Put $u = 2t \Rightarrow \int_1^2 \frac{1}{\ln 2t}\,dt = \int_2^4 \frac{1}{\ln u}\frac{du}{2} = \frac{1}{2}F(4).$