18.01a Exam 2, ESG Fall 2007 Solutions

Problem 1. (30 points)

A.) (10) Partial fractions: $\dfrac{x}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$

Coverup gives $A = 1/3$, $B = 2/3$.

$$\Rightarrow \boxed{\int \frac{x}{(x+1)(x-2)}\,dx = \frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|x-2| + C.}$$

B.) (10) Let $x = 2\sin u$. $\Rightarrow$ $dx = 2\cos u\,du$; $x = 0 \to u = 0$; $x = 2 \to u = \pi/2$.

$$\begin{aligned}\Rightarrow \text{ Integral } &= \int_0^{\pi/2} \frac{4\sin^2 u}{2\cos u}\, 2\cos u\,du \\ &= \int_0^{\pi/2} 4\sin^2 u\,du = 4\int_0^{\pi/2} \frac{1-\cos 2u}{2}\,du = 4\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Big|_0^{\pi/2} \\ &= \boxed{\pi}\end{aligned}$$

C.) (10) By parts: $\boxed{\begin{array}{ll} u = \sin^{-1} & dv = dx \\ du = \dfrac{dx}{\sqrt{1-x^2}} & v = x \end{array}}$

$$\Rightarrow \int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}\,dx} = \boxed{x\sin^{-1} x + (1-x^2)^{1/2} + C.}$$

Problem 2. (10 points)

$n = 4 \Rightarrow \Delta x = \dfrac{2}{4} = \dfrac{1}{2}$. Using the table at right.

j	0	1	2	3	4
x_j	1	3/2	2	5/2	3
y_j	1/2	4/13	1/5	4/29	1/10

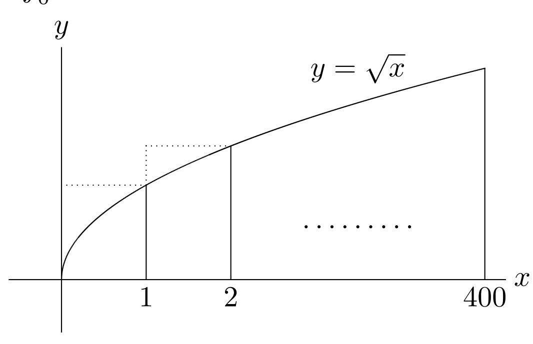
$$\begin{aligned}\int_1^3 \frac{1}{1+x^2}\,dx &\approx \left(\frac{y_0}{2} + y_1 + y_2 + y_3 + \frac{y_4}{2}\right)\Delta x \\ &= \boxed{(1/4 + 4/13 + 1/5 + 4/29 + 1/20)\cdot\frac{1}{2} = .473.}\end{aligned}$$

Problem 3. (15 points)

A. (10) Our sum is a right Riemann sum for $\int_0^{400} \sqrt{x}\,dx$ with $\Delta x = 1$ (see picture).

$$\Rightarrow \int_0^{400} \sqrt{x}\,dx \approx \sum_{j=1}^{400} \sqrt{j}.$$

$$\Rightarrow \sum_{j=1}^{400} \sqrt{j} \approx \frac{2}{3}x^{3/2}\Big|_0^{400} = \boxed{\frac{16000}{3}}.$$



B. (5) The RRS is an overestimate of the integral since $\sqrt{x}$ is increasing (see picture) $\Rightarrow$ the $\boxed{\text{integral is an underestimate of the sum.}}$

(continued)

Problem 4. (15 points)

A. (8) We use asymptotic comparison with $\dfrac{1}{x^2}$.

i. $\dfrac{x^2}{x^4+2x^2-1} \sim \dfrac{1}{x^2}$ (see this by inspection).

ii. $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$ converges by the p-test.

iii. Therefore, by asymptotic comparison, $\boxed{\displaystyle\int_2^\infty \frac{x^2}{x^4+2x^2-1}\,dx \text{ converges.}}$

B. (7) $\dfrac{e^{2x}}{x^3}$ grows to ∞ as x gets large.

Therefore, $\boxed{\displaystyle\int_1^\infty \frac{e^{2x}}{x^3}\,dx \text{ diverges}}$ i.e., the area $= \infty$.

Problem 5. (10 points)

A. (5) At $x = 2$ the integrand has a division by 0.

B. (5) We showed by direct computation in part 1C that the integral $\boxed{\text{converges.}}$

Problem 6. (10 points)

Slice into rings as shown. On each ring the density is constant so the number of holes in the ring $= dN = \delta(r)\,2\pi\,r\,dr$ (density times area).

$\Rightarrow$ total number of holes $= \displaystyle\int_0^\infty \frac{2\pi\,r}{1+r^p}\,dr = \int_0^1 \frac{2\pi\,r}{1+r^p}\,dr + \int_1^\infty \frac{2\pi\,r}{1+r^p}\,dr.$

The first integral is *not* improper $\Rightarrow$ it always converges.

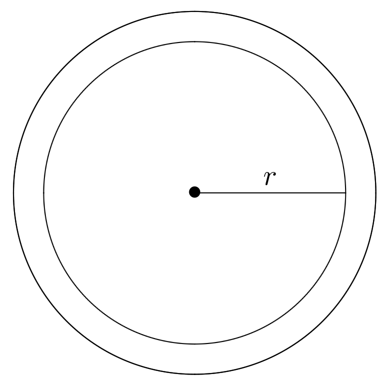
For the second integral we use asymptotic comparison to $\dfrac{1}{r^{p-1}}$:

i. $\dfrac{2\pi r}{1+r^p} \sim \dfrac{2\pi}{r^{p+1}}$ (see this by inspection).

ii. $\displaystyle\int_1^\infty \frac{2\pi}{4^{p-1}}\,dr$ converges for $p > 2$ (p-test).

iii. $\Rightarrow \displaystyle\int_1^\infty \frac{2\pi r}{1+r^p}\,dr$ converges for $p > 2$ (asymptotic comparison).

$\Rightarrow$ $\boxed{\text{for } p > 2 \text{ there are finitely many holes.}}$



Problem 7. (10 points)

By parts:

$$\boxed{\begin{array}{ll} u = (\ln x)^n & dv = x\,dx \\ du = n\dfrac{(\ln x)^{n-1}}{x}\,dx & v = x^2/2 \end{array}} \Rightarrow \boxed{\int x(\ln x)^n\,dx = \frac{x^2}{2}(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1}\,dx.}$$