18.01a Exam 3, ESG Fall 2007 Solutions

Problem 1. (25 points)

A. (9) Converges by asymptotic comparison to $\sum \frac{1}{n^{3/2}}$.

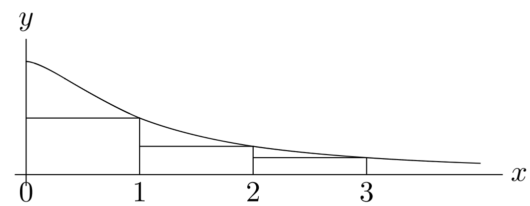
B. (8) Diverges: $\sum \frac{1}{\ln(n^3)} = \sum \frac{1}{3\ln n} > \frac{1}{3}\sum \frac{1}{n}$, which diverges.

C. (8)

Since $\frac{1}{x^2+1}$ is a decreasing function,

the right Riemann sum (with $\Delta x = 1$) for $\int_0^\infty \frac{1}{x^2+1}\,dx$

is an underestimate of the integral.



$$\Rightarrow \text{ right Riemann sum} = \sum_{n=1}^{\infty} \frac{1}{n^2+1} < \int_0^\infty \frac{1}{x^2+1}\,dx = \tan^{-1} x\Big|_0^\infty = \frac{\pi}{2}.$$

Problem 2. (10 points)

Let $g(x) = x - x^2 + x^3 - x^4 + x^5 - \ldots = \frac{x}{1+x}$. $\Rightarrow$ $f(x) = g'(x) = \frac{(1+x)-x}{(1+x)^2} = \boxed{\frac{1}{(1+x)^2}.}$

Problem 3. (15 points)

To be a density we need $\int_0^1 Ax^3\,dx = 1$. $\Rightarrow$ $A\frac{x^4}{4})\Big|_0^1 = A\cdot\frac{1}{4} = 1$ $\Rightarrow$ $\boxed{A = 4.}$

$$m = E(X) = \int_0^1 xf(x)\,dx = A\int_0^1 x\cdot x^3\,dx = A\,\frac{x^5}{5}\Big|_0^1 = A\cdot\frac{1}{5}. \;\Rightarrow\; \boxed{m = \frac{4}{5}.}$$

$$\sigma^2(X) = \int_0^1 x^2 f(x)\,dx - m^2 = A\int_0^1 x^2\cdot x^3\,dx - \frac{16}{25} = A\cdot\frac{1}{6} - \frac{16}{25} = \frac{4}{6} - \frac{16}{25} = \frac{4}{150} = \frac{2}{75}.$$

$\Rightarrow$ $\boxed{\sigma(X) = \sqrt{2/75}.}$

Problem 4. (10 points)

A. (5)

Function i:	Not a CDF –bigger than 1 in places.
Function ii:	Not a CDF –decreasing in places.
Function iii:	Is a CDF.
Function iv:	Not a CDF –decreasing in places.
Function v:	Is a CDF.

B. (5)

Function i:	Not a PDF –area > 1.
Function ii:	Could be a PDF.
Function iii:	Not a PDF –area > 1.
Function iv:	Not a PDF –area > 1.
Function v:	Not a PDF –area > 1.

(continued)

Problem 5. (15 points)

A. (7) Let X be the number of broken cookies in a box.
X is a Poisson r.v. with mean 3 $\Rightarrow$ $\boxed{P(X=0) = \mathrm{e}^{-3}.}$

B. (8) Let X_{10} be the number of broken cookies in 10 boxes.
X_{10} is a Poisson r.v. with mean 30.

$$P(X_{10} > 5) = 1 - P(X_{10} \leq 5) = \boxed{1 - \sum_{k=0}^{5} \mathrm{e}^{-30} \frac{30^k}{k!}.}$$

Problem 6. (15 points)

A. (5) Let T be the amount a random parent is late.
We're given T is an exponential r.v. with mean 5 minutes.
$\Rightarrow$ $\boxed{\text{Average lateness} = E(T) = 5.}$

B. (5) Density $= f(t) = \frac{1}{5}\mathrm{e}^{-t/5}$.

$$P(T > 15) = \int_{15}^{\infty} f(t)\,dt = \frac{1}{5}\int_{15}^{\infty} \mathrm{e}^{-t/5}\,dt = \mathrm{e}^{-t/5}\Big|_{15}^{\infty} = \boxed{\mathrm{e}^{-3}.}$$

C. (5) Average late fee $= \int t^2 f(t)\,dt = \frac{1}{5}\int_0^{\infty} t^2 \mathrm{e}^{-t/5}\,dt.$

The 'useful integral formula' on page 1 $\Rightarrow$ $\boxed{\text{average late fee} = \frac{1}{5}\frac{2!}{(1/5)^3} = 50 = \$.50.}$

Problem 7. (10 points)
Let X be the random variable representing weight. We need to find $P(X > 420)$.

Standardization $\Rightarrow$ $P(X > 420) = P(\frac{X-m}{\sigma} > \frac{420-m}{\sigma}) = P(Z > 1) = 1 - \Phi(1) = .16.$

$\Rightarrow$ we expect .16*100 = 16 apples in the box to weigh more than 420 g.

An alternative presentation: We know 420 is 1 standard deviation above the mean so
$P(X > 420) = P(Z > 1) = .16$ etc.