18.01a Practice Exam 1, ESG Fall 2007 Solutions

Problem 1. (8) $e^x \approx 1 + x + x^2/2$ and $\frac{1}{1+x} \approx 1 - x + x^2$
$\Rightarrow\ f(x) \approx (1 + x + x^2/2)(1 - x + x^2) \approx 1 + x^2/2.$

Problem 2. (7) $1 - e^{(x^2)} \approx -x^2$ and $\sin^2 x \approx x^2 \ \Rightarrow\ \lim_{x\to 0} \dfrac{1 - e^{(x^2)}}{\sin^2 x} = \lim_{x\to 0} \dfrac{-x^2}{x^2} = -1.$
Or by L'Hospital's rule twice (check indet. form 0/0 at each application):

$$\lim_{x\to 0} \frac{1 - e^{(x^2)}}{\sin^2 x} = \lim_{x\to 0} \frac{-2xe^{(x^2)}}{2\sin x\cos x} = \lim_{x\to 0} \frac{-4x^2 e^{(x^2)} - 2e^{(x^2)}}{2(\cos^2 x - \sin^2 x)} = -1$$

Problem 3. (15)
a) (5) Two forms (one is sufficient).
Slope form: If $f(x)$ is differentiable on $[a, b]$ then there is a c between a and b such that $\dfrac{f(b) - f(a)}{b - a} = f'(c).$
Analytic form: If $f(x)$ is differentiable on an interval with endpoints a and x then there is a c between a and x such that $f(x) = f(a) + f'(c)(x - a)$.
b) (10) $\tan(0) = 0$ and $\frac{d\tan x}{dx} = \sec^2 x \ \Rightarrow\ \tan x = (\sec^2 c)x$ for some c between 0 and x.
$\sec^2 c > 1$ for all c between 0 and $\pi/2 \ \Rightarrow\ \tan x > x$ for $0 < x < \pi/2$.

Problem 4. (10)
Method 1. $f'(x) = -2(1 + x)^{-3}, \quad f''(x) = 3 \cdot 2(1 + x)^{-4}, \quad f'''(x) = -4 \cdot 3 \cdot 2(1 + x)^{-5}$
$\Rightarrow\ f(0) = 1,\ f'(0) = -2,\ f''(0) = 6,\ f'''(0) = -24.$
$\Rightarrow\ f(x) = 1 - 2x + \dfrac{6}{2!}x^2 - \dfrac{24}{3!}x^3 + \ldots = 1 - 2x + 3x^2 + 4x^3 + \ldots.$
Method 2. Take $g(x) = \dfrac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 + \ldots.$
$\Rightarrow\ f(x) = -g'(x) = 1 - 2x + 3x^2 - 4x^3 + \ldots.$

Problem 5. (15) In the small time interval $[t, t + dt]$ the amount of material taken in is approximately $(1 - (t - 1)^2)\, dt$.
At the end of 2 years approx. $(1 - (t - 1)^2)\, dt\, e^{-k(2-t)}$ of this is left.
'Summing' this over the entire interval $[0, 2]$ gives: Total $= \int_0^2 (1 - (t - 1)^2) e^{-k(2-t)}\, dt$.

Problem 6. (10) Make the change of variables $u = \ln x$:
$du = \frac{1}{x}dx, \quad x = 2 \ \Rightarrow\ u = \ln 2, \quad x = 3 \ \Rightarrow\ u = \ln 3.$

$$\int_2^3 \frac{(1 + \ln x)^7}{x}\, dx = \int_{\ln 2}^{\ln 3} (1 + u)^7\, du = \frac{1}{8}(1 + u)^8 \Big|_{\ln 2}^{\ln 3}.$$

(continued)

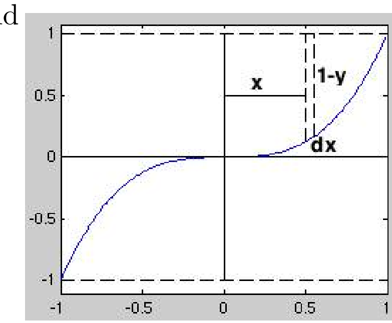
Problem 7. (20)

a) (10) The volume is symmetric so we do the top half and double it.

For the top half, $dV = 2\pi x(1-y)\,dx = 2\pi x(1-x^3)\,dx$.

Total volume of both halves:

$V = 2\int_0^1 2\pi x(1-x^3)\,dx = 2\pi(x^2 - \frac{2}{5}x^5)\Big|_0^1 = \frac{6}{5}\pi.$



b) (10) $ds = \sqrt{1+f'(x)^2}\,dx = \sqrt{1+9x^4}\,dx \;\Rightarrow\;$ Arclength $= \int ds = \int_{-1}^{1}\sqrt{1+9x^4}\,dx.$

Problem 8. (20)

a) (5) $F'(x) = \sqrt{3+\sin x},\quad F''(x) = \frac{1}{2}\frac{1}{\sqrt{3+\sin x}}\cos x$

$\Rightarrow\; F''(x) > 0$ for $0 < x < 1 \;\Rightarrow$ the graph is **Concave up.**

b) (5) $\sin x < 1 \;\Rightarrow\; F(1) = \int_0^1 \sqrt{3+\sin t}\,dt < \int_0^1 \sqrt{3+1}\,dt = 2.$

c) (5) Change of variable $u = 2t:\quad du = 2dt,\quad t = 1 \to u = 2,\quad t = 2 \to u = 4$

$\Rightarrow\; \int_1^2 \sqrt{3+\sin 2t}\,dt = \int_2^4 \sqrt{3+\sin u}\,du/2 = \frac{1}{2}F(x)\Big|_2^4 = \frac{F(4)-F(2)}{2}.$

d) (5) $G'(x) = (\sqrt{3+\sin x^2})2x.$

Problem 9. (15) a) (10) One hump has $0 \le x \le \pi$. The distance from the point on the graph $(x, \sin x)$ to the line $y = -1$ is $\sin x + 1$.

$\Rightarrow$ Ave. dist. $= \frac{1}{\pi}\int_0^\pi \sin x + 1\,dx = \frac{1}{\pi}\,(-\cos x + 1|_0^\pi = \frac{2+\pi}{\pi}.$

b) (5) The distance from the point on the graph $(x, \sin x)$ to the y-axis is x.

$\Rightarrow$ Ave. dist. $= \frac{1}{\pi}\int_0^\pi x\,dx = \frac{1}{\pi}\cdot\frac{x^2}{2}\Big|_0^\pi = \frac{\pi}{2}.$