18.01a Practice Exam 1b, ESG Fall 2007 Solutions

Problem 1. (14 points)

A. (7) $(1+x)^{3/5} \approx 1 + \frac{3}{5}x \Rightarrow (1.1)^{3/5} \approx \boxed{1.06.}$

B. (7) $(1+x)^{3/5} \approx 1 + \frac{3}{5}x - \frac{3}{25}x^2 \Rightarrow (1.1)^{3/5} \approx 1.06 - .0012 = \boxed{1.0588.}$

Problem 2. (20 points)

A. (7) We use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\infty/\infty}{=} \lim \frac{1/x}{1/(2\sqrt{x})} = \lim \frac{1}{2\sqrt{x}} = \boxed{0.}$$

B. (7) Some algebra followed by L'Hospital's rule:

Let $y = \lim_{x\to 0} x^x$. $\Rightarrow \ln y = \lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} \overset{0/0}{=} \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0.$

$\ln y = 0 \Rightarrow \boxed{y = 1.}$

C. (6) The best way to do this is algebraically.

Divide top and bottom by x^2 and the limit becomes $\lim_{x\to\infty} \frac{(1+\cos(x)/x^2)^2}{\sqrt{4 - 1/x}} = \boxed{\tfrac{1}{2}.}$

Problem 3. (6 points)

n	0	1	2	3
$f^{(n)}(x)$	$(1+x)^{3/5}$	$\frac{3}{5}(1+x)^{-2/5}$	$-\frac{6}{25}(1+x)^{-7/5}$	$\frac{42}{125}(1+x)^{-12/5}$
$f^{(n)}(0)$	1	$\frac{3}{5}$	$-\frac{6}{25}$	$\frac{42}{125}$

$$\Rightarrow \boxed{f(x) = 1 + \frac{3}{5}x - \frac{6/25}{2!}x^2 + \frac{42/125}{3!}x^3 + \ldots} = \boxed{f(x) = 1 + \frac{3}{5}x - \frac{3}{25}x^2 + \frac{7}{125}x^3 + \ldots}$$

Problem 4. (7 points) Make the change of variable $u = 1 + \cos x$.

$\Rightarrow du = -\sin x\, dx$ and $x = 0 \to u = 2$, $x = \pi/2 \to u = 1$.

Making all the substitutions the integral becomes

$$\int_2^1 -u^3\, du = \int_1^2 u^3\, du = \left.\frac{u^4}{4}\right|_1^2 = \boxed{4 - \frac{1}{4} = \frac{15}{4}.}$$

Problem 5. (7 points) The second fundamental theorem $\Rightarrow F'(x) = \frac{x^2-1}{e^t}$.

$\Rightarrow F'(x) = 0$ when $\boxed{x = \pm 1.}$

Problem 6. (6 points) Second fund. theorem and chain rule $\Rightarrow f(x^3)3x^2 = 2e^{2x}$

$\Rightarrow f(x^3) = \frac{2e^{2x}}{3x^2}$.

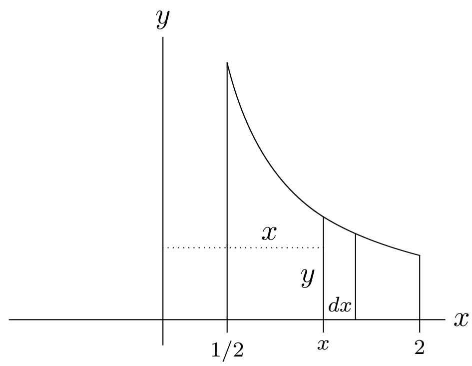
Replace x by $x^{1/3}$ $\Rightarrow \boxed{f(x) = \frac{2e^{2x^{1/3}}}{3x^{2/3}}.}$

(continued)

Problem 7. (20)
A. (10) We compute the volume by the method of shells.

The little slice pictured revolves into a cylindrical shell of volume $dV = 2\pi\, xy\, dx = 2\pi\, x \frac{1}{x}\, dx = 2\pi\, dx$.

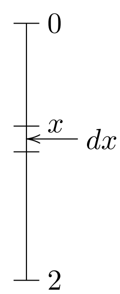
$\Rightarrow\ V = \int_{1/2}^{2} 2\pi\, dx = 2\pi(\frac{3}{2}) = \boxed{3\pi.}$



B. (10) $ds = \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \sqrt{1 + (1/x)^4}\, dx$. (Arclength element)

$\Rightarrow$ Surface area $= \int 2\pi x\, ds = \boxed{\int_{1/2}^{2} 2\pi x \sqrt{1 + (1/x)^4}\, dx.}$

Problem 8. (20)
A. (10) This is a 'slice and sum' problem.
We take a thin circular slice of thickness dx at depth x.
The radius of the slice is given by $r = 8 - 3x$.
The density of the slice is given by $\delta(x) = x^3 + 1$.
$\Rightarrow$ the mass of the slice is
$dm = \delta(x)\, \pi r^2\, dx = (x^3 + 1)(\pi(8 - 3x)^2)\, dx$.
$\Rightarrow$ total mass is $\boxed{M = \int_0^2 \pi(x^3 + 1)(8 - 3x)^2\, dx.}$



B. (10) Density $= \delta(x) = x^3 + 1$.

Ave. density $= \frac{1}{2}\int_0^2 \delta(x)\, dx = \frac{1}{2}\int_0^2 x^3 + 1\, dx = \frac{1}{2}\left(\frac{x^4}{4} + x\right)\Big|_0^2 = \boxed{3.}$

(Extra credit) (5) To lift the slice at depth x over the wall it must be lifted $x+1$ meters.
$\Rightarrow$ the work done on the slice is $dW = (x + 1)\, g\, dm$, where g is the acceleration of gravity.

$\Rightarrow$ Total work $= \boxed{\int_0^2 \pi g(x + 1)(x^3 + 1)(8 - 3x)^2\, dx.}$