18.01a Practice Exam 2, ESG Fall 2007 Solutions

Problem 1.

a) $\dfrac{7-x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$.

$x=1$, coverup $\Rightarrow$ $A = 6/2 = 3$.

Cross multiply: $7 - x = A(x^2+1) + (Bx+C)(x-1)$.

$x = 0 \Rightarrow 7 = A - C \Rightarrow C = -4$.

Coefficient of x^2: $0 = A - B \Rightarrow B = -3$.

$$\int \frac{7-x}{(x-1)(x^2+1)}\,dx = \int \frac{3}{x-1} - \frac{3x}{x^2+1} - \frac{4}{x^2+1}\,dx = 3\ln(x-1) - \frac{3}{2}\ln(x^2+1) - 4\tan^{-1}x + C.$$

b) Long division gives: $\dfrac{3x^3+6x^2+2x+2}{x^2+2x} = 3x + 2\dfrac{x+1}{x(x+2)}$.

Coverup gives: $2\dfrac{x+1}{x(x+2)} = \dfrac{1}{x} + \dfrac{1}{x+2}$.

$$\Rightarrow \int \frac{3x^3+6x^2+2x+2}{x^2+2x}\,dx = \frac{3}{2}x^2 + \ln x + \ln(x+2) + C.$$

c) By parts:

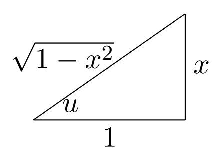
$$\begin{aligned}\int \sin^{-1}x\,dx &= x\sin^{-1}x - \int \tfrac{x}{\sqrt{1-x^2}}\,dx \\ &= x\sin^{-1}x - \sqrt{1-x^2} + C.\end{aligned}$$

$u = \sin^{-1}x$	$dv = dx$
$du = \frac{1}{\sqrt{1-x^2}}\,dx$	$v = x$

d) Let $x = \tan u$, $dx = \sec^2 u\,du$:

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{\sec^2 u}{\sec^4 u}\,du = \int \cos^2 u\,du = \int \frac{1+\cos 2u}{2}\,du$$

$$= \frac{u}{2} + \frac{\sin 2u}{4} + C = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2} + C.$$



e) Let $x = 2\sin u \Rightarrow dx = 2\cos u\,du$; $x = 0 \to u = 0$, $x = 1 \to u = \sin^{-1}(1/2) = \pi/6$.

$$\begin{aligned}\Rightarrow \text{ integral } &= \int_0^{\pi/6} \frac{4\sin^2 u}{\sqrt{4-4\sin^2 u}}\,2\cos u\,du = \int_0^{\pi/6} 4\sin^2 u\,du \\ &= \int_0^{\pi/6} 4\frac{1-\cos 2u}{2}\,du = \frac{4u}{2} - \frac{4\sin 2u}{4}\Big|_0^{\pi/6} \\ &= \frac{\pi}{3} - \sin(\pi/3) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}.\end{aligned}$$

Problem 2.

a) $n = 2$, $\Delta x = \pi/2$.

Table:

j	0	1	2
x_j	0	$\pi/2$	π
y_j	1	2	1

$$\int f(\theta)\,d\theta \approx \frac{1}{3}(y_0 + 4y_1 + y_2)\Delta x = \frac{1}{3}(1+8+1)\frac{\pi}{2} = \frac{5}{3}\pi.$$

(continued)

b) $f(t) = 1/t$, $a = 1$, $b = N$, $n = N-1$, $\Delta x = 1$.
Trapezoidal rule

$$\Rightarrow \int_1^N \frac{1}{t}\,dt \approx (\frac{1}{2}\cdot 1 + \frac{1}{2} + \dots \frac{1}{N-1} + \frac{1}{2}\cdot\frac{1}{N})$$

$$\Rightarrow \int_1^N \frac{1}{t}\,dt + \frac{1}{2} + \frac{1}{2N} \approx (1 + \frac{1}{2} + \dots \frac{1}{N-1} + \frac{1}{N})$$

$$\Rightarrow \ln N + \frac{1}{2} + \frac{1}{2N} \approx (1 + \frac{1}{2} + \dots \frac{1}{N-1} + \frac{1}{N}).$$

Table:

j	0	1	$\dots$	$N-1$
x_j	1	2	$\dots$	N
y_j	1	$\frac{1}{2}$	$\dots$	$\frac{1}{N}$

Since $f(t)$ is concave up the trapezoidal rule gives an overestimate for the integral:

$$\Rightarrow \int_1^N \frac{1}{t}\,dt < (\frac{1}{2}\cdot 1 + \frac{1}{2} + \dots \frac{1}{N-1} + \frac{1}{2}\cdot\frac{1}{N})\cdot 1$$

$$\Rightarrow \ln N + \frac{1}{2} + \frac{1}{2N} < (1 + \frac{1}{2} + \dots \frac{1}{N-1} + \frac{1}{N}).$$

So we have an underestimate for the sum.

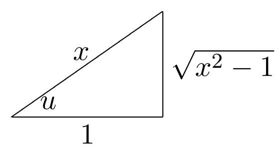
Problem 3.

a) By parts: $\displaystyle\int x^2 \ln x\,dx = \frac{x^3}{3} - \int \frac{x^2}{3}\,dx = \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C.}$

$u = \ln x$	$dv = x^2\,dx$
$du = \frac{1}{x}\,dx$	$v = \frac{x^3}{3}$

b) By parts: $\displaystyle\int \sec^{-1} x\,dx = x\sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}}\,dx.$

$u = \sec^{-1} x$	$dv = dx$
$du = \dfrac{1}{x\sqrt{x^2-1}}\,dx$	$v = x$



To compute the integral on the right hand side let $x = \sec u$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \int \frac{\sec u \tan u}{\tan u}\,du = \int \sec u\,du = \ln(\sec u + \tan u) + C = \ln(x + \sqrt{x^2-1}) + C.$$

$$\Rightarrow \int \sec^{-1} x\,dx = x\sec^{-1} x - \ln(x + \sqrt{x^2-1}) + C.$$

c) By parts:

$u = \sin^{n-1} x$	$dv = \sin x\,dx$
$du = -(n-1)\sin^{n-2} x\cos x\,dx$	$v = -\cos x$

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}\cos^2 x\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}(1-\sin^2 x)\,dx \\ &= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}\,dx - (n-1)\int \sin^2 x\,dx\end{aligned}$$

$$\Rightarrow n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}\,dx$$

$$\Rightarrow \boxed{\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2}\,dx}$$

(continued)

$$\begin{aligned}
\int \sin^6 \, dx &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 x\,dx \\
&= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{6}\cdot\frac{1}{4}\sin^3 x\cos x + \frac{5}{6}\cdot\frac{3}{4}\int \sin^2 x\,dx \\
&= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{6}\cdot\frac{1}{4}\sin^3 x\cos x - \frac{5}{6}\cdot\frac{3}{4}\frac{1}{2}\sin x\cos x + \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int dx \\
&= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{6}\cdot\frac{1}{4}\sin^3 x\cos x - \frac{5}{6}\cdot\frac{3}{4}\frac{1}{2}\sin x\cos x + \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}x.
\end{aligned}$$

$$\Rightarrow \boxed{\int_0^{\pi/2} \sin^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}.}$$

Problem 4. By definition $\displaystyle\int_0^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx.$

By parts: $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}.$ $\boxed{\begin{array}{ll} u = \ln x & dv = \frac{1}{x^2}\,dx \\ du = \frac{1}{x}\,dx & v = -\frac{1}{x}\end{array}}$

$$\Rightarrow \int_1^b \frac{\ln x}{x^2}\,dx = -\frac{\ln b}{b} - \frac{1}{b} + 1 \to 1 \text{ as } b\to\infty. \text{ (So the answer is 1.)}$$

Problem 5.

a) Limit compare with $\int \frac{1}{\sqrt{x}}$ (which diverges):

Ratio $= \dfrac{x/\sqrt{x^3+1}}{1/\sqrt{x}} = \dfrac{x^3/2}{\sqrt{x^3+1}} \to 1$ as $x\to\infty$.

$\Rightarrow$ both integrals behave the same $\Rightarrow$ integral diverges.

b) Limit compare with $\sum \frac{1}{n}$ (which diverges):

Ratio $= \dfrac{(n^2+1)/(n^3+1)}{1/n} = \dfrac{n^3+n}{n^3+1} \to 1.$ $\Rightarrow$ series behave the same $\Rightarrow$ sum diverges.

c) Integral test: $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = -\frac{1}{\ln x}\Big|_2^\infty = \frac{1}{\ln 2}.$

The integral converges $\Rightarrow$ the sum converges.

Problem 6.

$$\begin{aligned}
\int_0^\infty \frac{1}{1+t^4}\,dt &= \int_0^1 \frac{1}{1+t^4}\,dt + \int_1^\infty \frac{1}{1+t^4}\,dt \\
&< \int_0^1 1\cdot dt + \int_1^\infty \frac{1}{t^4}\,dt = 1 - \frac{1}{3x^3}\Big|_1^\infty = 1 + \frac{1}{3} = \frac{4}{3}.
\end{aligned}$$

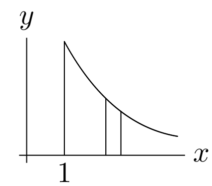
(continued)

Problem 7.

a) We know $\int_0^\infty \frac{1}{x^p}\,dx = \begin{cases} \infty & \text{for } p \le 1 \\ \frac{1}{p-1} & \text{for } p > 1 \end{cases}$ (You should be able to derive this.)

Volume by shells: $dV = \pi y^2\,dx = \pi \frac{1}{x^{2p}}\,dx.\ \Rightarrow\ V = \int_1^\infty \pi \frac{1}{x^2 p}\,dx$

Volume is finite for $2p > 1 \ \Rightarrow\ p > 1/2$



Surface area: $ds = \sqrt{1 + (y')^2}\,dx = \sqrt{1 + p^2 x^{-2(p+1)}}\,dx$

$dA = 2\pi y\,ds = 2\pi \frac{1}{x^p}\sqrt{1 + p^2 x^{-2(p+1)}}\,dx$

$A = \int_1^\infty 2\pi \frac{1}{x^p}\sqrt{1 + p^2 x^{-2(p+1)}}\,dx$

Limit compare with $\frac{1}{x^p}$ $\Rightarrow$ area is infinite if $p \le 1$.

Thus, volume is finite and surface area is infinite for $1/2 < p \le 1$.

b) There is no problem. A finite volume sliced into thin enough pieces (infinitesimally thick) can cover an infinite surface. Of course, given the finite size of the paint molecules, I can't really fill the volume: the molecules can't get into the narrow part of the volume.