18.01a Practice Exam 3, ESG Fall 2007 Solutions

Problem 1. We have to do some integrals.
Total probability = 1 = \int_0^\pi C \sin x \, dx = -C \cos x|_0^\pi = 2C. \quad \Rightarrow \quad C = 1/2.

\[ m = E(X) = \int_0^\pi C f(x) \, dx = \int_0^\pi C x \sin x \, dx = C \pi = \pi/2. \]

\[ \sigma^2(X) = \int_0^\pi x^2 f(x) \, dx - m^2 = \int_0^\pi C x^2 \sin x \, dx - \pi^2/4 = \pi^2/4 - 2 = .467 \Rightarrow \sigma = .68. \]
(The integrals for \( m \) and \( \sigma \) are easy to compute by parts.)

Problem 2.

a) Let \( X \) = number of errors on 9 pages.
\( X \) is Poisson with mean \( m = 9 \cdot \frac{200}{300} = 6. \) \( \Rightarrow \) 
\[ P(X = 0) = e^{-m} = e^{-6} = 0.002 = 2\%. \]

b) Let \( X \) = number of errors on 4 pages. \( X \) is Poisson with mean \( m = 4 \cdot \frac{200}{300} = \frac{8}{3}. \)
\( \Rightarrow P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-m} - e^{-m}m = 1 - e^{-2.67} - 2.67e^{-2.67} = .75\%.

Problem 3. Let \( X \) be the time between accidents. \( X \) is exponential with mean 10 hours.
\( \Rightarrow \) Density function \( f(x) = \frac{1}{10} e^{-x/10}. \)
\( \Rightarrow P(X > 24) = \int_{24}^{\infty} f(x) \, dx = \int_{24}^{\infty} \frac{1}{10} e^{-x/10} \, dx = e^{-24/10} = 9\%. \)

Problem 4. Standardize:
\( P(84 < X < 120) = P(-16/16 < \frac{X - m}{\sigma} < 20/16) = \Phi(1.25) - \Phi(-1) = \Phi(1.25) + \Phi(1) - 1. \)
Table lookup \( \Rightarrow P(84 < X < 120) \approx .89 + .84 - 1 = .73. \)
We would expect \( \frac{.73 \cdot 144 = 105 \text{ bulbs}}{} \) to last in the range 84 to 120 hours?

Problem 5. a) Let \( f(x) = \frac{x^3}{3} + \frac{x^5}{5} + \ldots \) \( \Rightarrow f'(x) = 1 + x^2 + x^4 + \ldots = \frac{1}{1 - x^2}. \)
(The last equality is the sum of a geometric series.)
\( \Rightarrow f(x) = \int \frac{1}{1 - x^2} \, dx = (\text{partial fractions}) \int \frac{1/2}{1 + x} + \frac{1/2}{1 - x} \, dx = \ln \left( \frac{1 + x}{1 - x} \right) + C. \)

Since \( f(0) = 0 \) we get \( C = 0 \) \( \Rightarrow f(x) = \ln \left( \frac{1 + x}{1 - x} \right). \)

b) Let \( f(x) = x + 2x^2 + 3x^3 + 4x^4 + \ldots \)
\( f(x) = x(1 + 2x + 3x^2 + 4x^3 + \ldots) = x \frac{d}{dx} (x + x^2 + x^3 + \ldots) = x \frac{d}{dx} \frac{x}{1 - x} = \frac{x}{(1 - x)^2}. \)

Problem 6.
a) Limit compare with \( \sum \frac{1}{n} \) (which diverges):
\( \text{Ratio} = \frac{(n^2 + 1)/(n^3 + 1)}{1/n} = \frac{n^3 + n}{n^3 + 1} \rightarrow 1. \) \( \Rightarrow \) series behave the same \( \Rightarrow \) sum diverges.

b) Integral test: \( \int_2^\infty \frac{1}{x(\ln x)^2} \, dx = - \frac{1}{\ln x}\big|_2^\infty = \frac{1}{\ln 2} \). Integral converges \( \Rightarrow \) sum converges.