18.01a Practice Exam 3, ESG Fall 2007 Solutions

Problem 1. We have to do some integrals.

Total probability $= 1 = \displaystyle\int_0^\pi C\sin x\,dx = -C\cos x|_0^\pi = 2C. \Rightarrow \boxed{C = 1/2.}$

$m = E(X) = \displaystyle\int x f(x)\,dx = \int_0^\pi Cx\sin x\,dx = C\pi = \boxed{\pi/2.}$

$\sigma^2(X) = \displaystyle\int x^2 f(x)\,dx - m^2 = \int_0^\pi Cx^2\sin x\,dx - \pi^2/4 = \boxed{\pi^2/4 - 2 = .467 \Rightarrow \sigma = .68.}$

(The integrals for m and σ are easy to compute by parts.)

Problem 2.

a) Let X = number of errors on 9 pages.

X is Poisson with mean $m = 9\cdot\frac{200}{300} = 6. \Rightarrow \boxed{P(X=0) = \mathrm{e}^{-m} = \mathrm{e}^{-6} = .002 = 2\%.}$

b) Let X = number of errors on 4 pages. X is Poisson with mean $m = 4\cdot\frac{200}{300} = \frac{8}{3}$.

$\Rightarrow P(X\ge 2) = 1 - P(X=0) - P(X=1) = \boxed{1 - \mathrm{e}^{-m} - \mathrm{e}^{-m}m = 1 - \mathrm{e}^{-2.67} - 2.67\mathrm{e}^{-2.67} = 75\%.}$

Problem 3. Let X be the time between accidents. X is exponential with mean 10 hours.

$\Rightarrow$ Density function $f(x) = \frac{1}{10}\mathrm{e}^{-x/10}$.

$\Rightarrow P(X > 24) = \displaystyle\int_{24}^\infty f(x)\,dx = \int_{24}^\infty \frac{1}{10}\mathrm{e}^{-x/10}\,dx = \boxed{\mathrm{e}^{-24/10} = 9\%.}$

Problem 4. Standardize:

$P(84 < X < 120) = P(-16/16 < \dfrac{X-m}{\sigma} < 20/16) = \Phi(1.25) - \Phi(-1) = \Phi(1.25) + \Phi(1) - 1.$

Table lookup $\Rightarrow \boxed{P(84 < X < 120) \approx .89 + .84 - 1 = .73.}$

We would expect $\boxed{.73\cdot 144 = 105 \text{ bulbs}}$ to last in the range 84 to 120 hours?

Problem 5. a) Let $f(x) = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots \Rightarrow f'(x) = 1 + x^2 + x^4 + \ldots = \dfrac{1}{1-x^2}$.

(The last equality is the sum of a geometric series.)

$\Rightarrow f(x) = \displaystyle\int \frac{1}{1-x^2}\,dx = \text{(partial fractions)} \int \frac{1/2}{1+x} + \frac{1/2}{1-x}\,dx = \ln\sqrt{\frac{1+x}{1-x}} + C.$

Since $f(0) = 0$ we get $C = 0 \Rightarrow f(x) = \boxed{\ln\sqrt{\dfrac{1+x}{1-x}}.}$

b) Let $f(x) = x + 2x^2 + 3x^3 + 4x^4 + \ldots$.

$f(x) = x(1 + 2x + 3x^2 + 4x^3 + \ldots) = x\dfrac{d}{dx}(x + x^2 + x^3 + \ldots) = x\dfrac{d}{dx}\dfrac{x}{1-x} = \boxed{\dfrac{x}{(1-x)^2}.}$

Problem 6.

a) Limit compare with $\sum \dfrac{1}{n}$ (which diverges):

Ratio $= \dfrac{(n^2+1)/(n^3+1)}{1/n} = \dfrac{n^3+n}{n^3+1} \to 1. \Rightarrow$ series behave the same $\Rightarrow$ sum diverges.

b) Integral test: $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = -\frac{1}{\ln x}\Big|_2^\infty = \frac{1}{\ln 2}.$ Integral converges $\Rightarrow \boxed{\text{sum converges.}}$