18.02a Practice Exam 4, ESG Fall 2007 Solutions

Problem 1.

a) $\overrightarrow{\mathbf{PA}} = \langle 1,1,0\rangle - \langle 0,0,2\rangle = \langle 1,1,-2\rangle.$
$\overrightarrow{\mathbf{PB}} = \langle 1,-1,0\rangle - \langle 0,0,2\rangle = \langle 1,-1,-2\rangle.$

b) Want $\theta = \angle APB:\ \overrightarrow{\mathbf{PA}}\cdot\overrightarrow{\mathbf{PB}} = |\overrightarrow{\mathbf{PA}}||\overrightarrow{\mathbf{PB}}|\cos\theta = 4.\ \Rightarrow\ \boxed{\theta = \cos^{-1}(4/6).}$
(with a calculator: $\theta = 48°$)

c) Area $= \frac{1}{2}|\overrightarrow{\mathbf{PA}}\times\overrightarrow{\mathbf{PB}}|$.

$$\overrightarrow{\mathbf{PA}}\times\overrightarrow{\mathbf{PB}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 1 & -1 & -2 \end{vmatrix} = -4\mathbf{i} - 2\mathbf{k}\ \Rightarrow\ \boxed{\text{area} = \tfrac{1}{2}\sqrt{20} = \sqrt{5}.}$$

Problem 2.

a) Determinant computed using top row of cofactors: $|A| = 1\cdot 3 + 0\cdot(-5) + 1\cdot 1 = 4.$
We compute minors $\rightarrow$ cofactors $\rightarrow$ adjoint $\rightarrow A^{-1}$.

$$\begin{pmatrix} 3 & 5 & 1 \\ -1 & 1 & 1 \\ -2 & -2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & -1 \\ -2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 & -2 \\ -5 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix} \rightarrow \boxed{A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -2 \\ -5 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix}.}$$

(Also, I checked that $A\cdot A^{-1} = I$.)

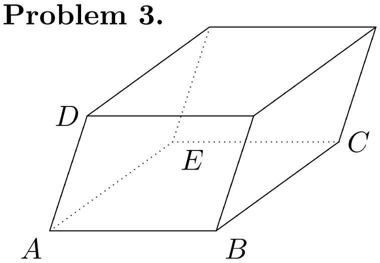
b) The system is $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}\ \Rightarrow\ \mathbf{x} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \boxed{\frac{1}{4}\begin{pmatrix} -5 \\ 3 \\ 9 \end{pmatrix}.}$

c) There are non-zero solutions to the homogeneous solution when the determinant of the coefficient matrix is 0. $\Rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow 2c = 0 \Rightarrow \boxed{c = 0.}$

d) $\overrightarrow{\mathbf{A}} = \langle 1,0,1\rangle$, $\overrightarrow{\mathbf{B}} = \langle 3,2,1\rangle$ and $\overrightarrow{\mathbf{C}} = \langle 1,1,0\rangle$ are the rows of the matrix.
Determinant equal 0 means they all lie in a plane (volume of parallelpiped = 0)

$$\Rightarrow \text{Normal to plane} = \overrightarrow{\mathbf{A}}\times\overrightarrow{\mathbf{B}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\ \Rightarrow\ \boxed{\textbf{Answer}:\ (-2,2,2).}$$

Problem 3.



Compute volume using determinant with rows $\overrightarrow{\mathbf{AB}}$, $\overrightarrow{\mathbf{AE}}$, $\overrightarrow{\mathbf{AD}}$.
$\overrightarrow{\mathbf{AE}} = \overrightarrow{\mathbf{BC}} = \langle 3,2,1\rangle$, $\overrightarrow{\mathbf{AB}} = \langle 1,0,1\rangle$, $\overrightarrow{\mathbf{AD}} = \langle 1,1,2\rangle$

$$\Rightarrow \pm\text{vol} = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -4\ \Rightarrow\ \boxed{\textbf{Answer}:\ 4.}$$

(continued)

Problem 4.

a) Let $A = (1, 1, 1)$, $B = (1, 2, 1)$, $C = (2, 2, 3)$ $\Rightarrow$ $\overrightarrow{\mathbf{AB}} = \langle 0, 1, 0\rangle$ and $\overrightarrow{\mathbf{AC}} = \langle 1, 1, 2\rangle$.

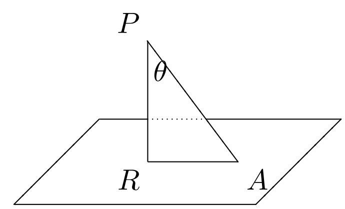
$$\text{Normal to plane} = \overrightarrow{\mathbf{N}} = \overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2\mathbf{i} - \mathbf{k} = \langle 2, 0, -1\rangle.$$

$\Rightarrow$ Plane (point-normal form): $2(x-1) + 0\cdot(y-1) - (z-1) = 0 \Leftrightarrow \boxed{2x - z = 1.}$

b)

$P = (0, 0, 3)$

$$\Rightarrow |\mathbf{PR}| = \left|\overrightarrow{\mathbf{PA}} \cdot \frac{\overrightarrow{\mathbf{N}}}{|\overrightarrow{\mathbf{N}}|}\right| = \left|\langle 1, 1, -2\rangle \cdot \frac{\langle 2, 0, -1\rangle}{\sqrt{5}}\right| = \boxed{\frac{4}{\sqrt{5}}.}$$



Problem 5. $x = 2 + t$, $y = 3 + 3t$, $z = 5t$.

Point (x, y, z) on plane $\Rightarrow$ $2(2 + t) - 3(3 + 3t) + 5t = 7 \Rightarrow t = -6$

$\Rightarrow$ $\boxed{(x, y, z) = (-4, -15, -30).}$ (Check: -8+45-30 = 7)

Problem 6.

a) $x = t$ and $y = \sin t \Rightarrow \mathbf{r}(t) = t\,\mathbf{i} + \sin t\,\mathbf{j} = \langle t, \sin t\rangle$.

b) $\dfrac{d\mathbf{r}}{dt} = \boxed{\mathbf{i} + \cos t\,\mathbf{j}.}$

$$\Rightarrow \frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \boxed{\sqrt{1 + \cos^2 t}.} \qquad \mathbf{T}(t) = \frac{d\mathbf{r}/dt}{ds/dt} = \boxed{\frac{1}{\sqrt{1+\cos^2 t}}(\mathbf{i} + \cos t\,\mathbf{j}).}$$

$$\frac{d\mathbf{T}}{dt} = \frac{\cos t \sin t}{(1 + \cos^2 t)^{3/2}}\langle 1, \cos t\rangle + \frac{1}{(1 + \cos^2 t)^{1/2}}\langle 0, -\sin t\rangle = -\frac{\sin t}{(1 + \cos^2 t)^{3/2}}\langle \cos t, -1\rangle$$

$$\Rightarrow \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = -\frac{\sin t}{(1 + \cos^2 t)^2}(\cos t, -1) \Rightarrow \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \boxed{\frac{|\sin t|}{(1 + \cos^2 t)^{3/2}}.}$$

(Or could use the formula: $\kappa = \dfrac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \dfrac{|(-\sin t\,\mathbf{j}) \times (\mathbf{i} + \cos t\,\mathbf{j})|}{|\mathbf{v}|^3} = \dfrac{|\sin t\,\mathbf{k}|}{(1 + \cos^2 t)^3} = \dfrac{|\sin t|}{(1 + \cos^2 t)^3}$.)

Problem 7. The graph shows a 'rounded' cycloid

The parameter θ is the angle through which the circle has rolled.

$\mathbf{r}(\theta) = \overrightarrow{\mathbf{OP}} = \overrightarrow{\mathbf{OR}} + \overrightarrow{\mathbf{RC}} + \overrightarrow{\mathbf{CP}}$.

$\overrightarrow{\mathbf{OR}} = a\theta\,\mathbf{i}$.

$\overrightarrow{\mathbf{RC}} = a\,\mathbf{j}$.

$\overrightarrow{\mathbf{CP}} = \frac{1}{2}\overrightarrow{\mathbf{CQ}} = -\frac{a}{2}(\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$.

$\Rightarrow$ $\boxed{\mathbf{r}(\theta) = a(\theta - \tfrac{1}{2}\sin\theta)\,\mathbf{i} + a(1 - \tfrac{1}{2}\cos\theta)\,\mathbf{j}.}$

or $\boxed{x = a(\theta - \tfrac{1}{2}\sin\theta), \quad y = a(1 - \tfrac{1}{2}\cos\theta).}$