

## 18.02a Practice Midterm Questions, Fall 2007

**Problems 1-5 cover material from the first unit. This will be on the midterm, but won't be emphasized.**

Problems 1-5 take about 1-1.5 hours, problems 6-19 take 2-3 hours.

The actual test will be shorter –designed to take 2 hours, with simpler arithmetic.

**Problem 1.** Consider the point  $P = (20, 0, 0)$ , the plane  $\mathcal{P} : x + 2y + 3z = 6$ , and the point  $Q = (1, 1, 1)$  on  $\mathcal{P}$ .

- Compute the distance from  $P$  to  $\mathcal{P}$ .
- Give parametric equations for the line through  $P$  and perpendicular to  $\mathcal{P}$ .
- Find the point of intersection between  $\mathcal{P}$  and the line of part(b). For later reference, call this point  $R$ .
- Find the angle,  $\angle PQR$ .
- By computing  $|PR|$  directly, verify your answer to part (a).
- Find the area of the triangle with vertices  $P$ ,  $Q$  and  $R$ .

**Problem 2.** Suppose tape is unwound from a roll in such a way that it is always vertical. Assuming the roll is centered at the origin and has radius 2, and the end of the tape starts at the point  $(2, 0)$ , give parametric equations for the path traced out by the end of the roll. For what values of your parameter does this make sense?

**Problem 3.** The motion of a point  $P$  is given parametrically by

$$\overrightarrow{OP} = \mathbf{r}(t) = \langle 4 \sin t, 5 \cos t, 3 \sin t \rangle.$$

- Find  $\mathbf{v}$ ,  $\frac{ds}{dt}$ ,  $\mathbf{T}$ ,  $\kappa$ .
- Show  $\mathbf{r}$  is perpendicular to  $3\mathbf{i} - 4\mathbf{k}$ . What information about the motion of the point  $P$  does this give?

**Problem 4.** Let  $(\sin t, \cos 2t)$  be a parametrized path of a point  $P$  in the plane. Give the  $x$ - $y$  equation of this path. Sketch and describe how  $P$  moves over time.

**Problem 5.** Let  $A_c = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & c \end{pmatrix}$ .

- Let  $B = \frac{1}{4} \begin{pmatrix} 3 & 1 & -2 \\ -5 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix}$ .

Show  $B$  is the inverse of  $A_2$  (the subscript indicates  $c = 2$ ). Show your arithmetic carefully.

- Use part (a) to solve  $x + z = 1$ ,  $3x + 2y + z = 0$ ,  $x + y + 2z = 4$ .
- For what  $c$  will the system of equations  $A_c \mathbf{x} = \mathbf{0}$  have a non-zero solution?
- For the value of  $c$  found in part (c) find a non-zero solution to the system.
- Compute  $A_1^{-1}$ .

*(continued)*

**Problem 6.**

- a) Find the normal to the level surface  $x^3 + y^3z = 3$  at the point  $(1, 1, 2)$ .
- b) Use level surfaces to show the perpendicular to the graph of  $z = f(x, y)$  is given by  $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \rangle$ . (You can use what you know about gradients and level surfaces.)

**Problem 7.** Graph the surface and level curves of  $z = y^2 - x$ .

**Problem 8.** Let  $w = f(x, y)$  and  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ . Along the path given by  $\mathbf{r}$  we have  $w = f(x(t), y(t))$ .

- a) Assuming everything is differentiable, show  $\frac{dw}{dt} = \nabla w \cdot \frac{d\mathbf{r}}{dt}$ .
- b) Suppose the path is along a level curve of  $f$ . Show that  $\nabla w$  is perpendicular to the level curve.

**Problem 9.** Suppose  $w = f(xy)$  (i.e.  $w = f(u)$  with  $u = xy$ ). This implies  $x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y} = 0$ .

- a) Verify this for the function  $w = \sin(xy)$ .
- b) Show this in general.

**Problem 10.** Suppose  $w = x^2 + y^2 + z^2$  and  $x, y, z$  are related by  $x = f(y, z)$ .

- a) Suppose  $f(y, z) = yz$ , find  $(\frac{\partial w}{\partial z})_x$ . Do this twice, once with the chain rule and once with differentials. Do not do it by solving for  $y$  in terms of  $z$  (except to check your answer).
- b) For arbitrary  $f$ , Find  $(\frac{\partial w}{\partial z})_x$  in terms of the formal partials of  $f(y, z)$ . (Again, do this twice, once with the chain rule and once with differentials.)

**Problem 11.**

- a) Starting from the basic chain rule:  $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}$ , derive the matrix equation relating  $(\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v})$  and  $(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$
- b) Write down the matrix equation in the case where  $(x, y)$  is as usual and  $(u, v) = (r, \theta)$  (polar coordinates).
- c) Same question for  $(u, v) = (x, \theta)$ .

**Problem 12.** Let  $w = x^3y + x/y$  and  $P = (2, 1)$ .

- a) Compute the gradient  $\nabla w|_P$ .
- b) Compute the directional derivative,  $\frac{dw}{ds}$  at  $P$  in the direction of  $\mathbf{i} + 3\mathbf{j}$ .
- c) Find a direction at  $P$  in which  $w$  is not changing.
- d) Estimate the value of  $w$  at  $(2.1, 0.9)$ .

**Problem 13.** Give the equation for the tangent plane to the surface  $x^2 + xy^2 + yz^2 = 23$  at the point  $(1, 2, 3)$ .

(continued)

**Problem 14.** Suppose you have an open box of volume 4 with dimensions  $x, y, z$ . So we all use the same notation, assume the open end is one of the sides with dimensions  $y$  and  $z$ .

- By substituting for  $z$  write down the unconstrained equation for the surface area of the box.
- Use part (a) to find the dimensions that minimize the area.
- Use the second derivative test to verify your answer to part (b) is a minimum
- In part (a)  $x$  and  $y$  can be anywhere in a region  $R$ . Describe  $R$ . What is its boundary?
- Why can't the minimum area occur on the boundary?
- Redo the minimization using Lagrange multipliers.

**Problem 15.** Find the critical points of  $x^2 - 2xy^2 + 2y^2$ . Classify them as minima, maxima or saddle points.

**Problem 16.** Evaluate by reversing the limits of integration:  $\int_0^1 \int_{x^{\frac{1}{3}}}^1 e^{y^4} dy dx$ .

**Problem 17.** Let  $R$  be a circular region of radius  $a$  and uniform density. Set up (**but do not evaluate**) iterated integrals in polar coordinates for the following moments of inertia. You can make things easier by carefully choosing where to put  $R$  for each problem.

- Moment of inertia of  $R$  about its center.
- Moment of inertia of  $R$  about a point on its edge.
- Moment of inertia of  $R$  about a diameter.
- Moment of inertia of  $R$  about a line tangent to the circle.
- Let  $S$  be the region in the first quadrant, bounded below by the  $x$ -axis, above by  $y = x$  and on the right by the circle of radius 1 with center at  $(1, 0)$  (NOTE CENTER). Assume uniform density and write down (but don't evaluate) an integral in polar coordinates for the polar moment of inertia (i.e. moment of inertia about the origin) of  $S$ .

**Problem 18.** Find the mass of the planar region inside the cardioid  $r = a(1 + \cos \theta)$  with density given by  $\delta(r, \theta) = 1/r$ .

**Problem 19.** Evaluate  $\iint_R (2x - 3y)^2 (x + y)^2 dx dy$ , where  $R$  is the triangle bounded by the positive  $x$ -axis, negative  $y$ -axis and the line  $y = \frac{2}{3}x - \frac{4}{3}$ , by making a change of variable  $u = x + y, v = 2x - 3y$ .