18.01A Problem Set 4 Part I Solutions

For a number of these solutions we use the following notes:

N1. $\displaystyle\int \frac{\mathrm{e}^{-t/m}}{m}\,dt = -\mathrm{e}^{-t/m}.$

N2. $\displaystyle\int t\frac{\mathrm{e}^{-t/m}}{m}\,dt = -t\mathrm{e}^{-t/m} - m\mathrm{e}^{-t/m}.$

N3. $\displaystyle\int t^2\frac{\mathrm{e}^{-t/m}}{m}\,dt = -t^2\mathrm{e}^{-t/m} - 2mt\mathrm{e}^{-t/m} - 2m^2\mathrm{e}^{-t/m}.$

N4. $\displaystyle\lim_{t\to\infty} t^k\mathrm{e}^{-mt} = 0$ for any k.

8A-1

a) $P(X \text{ is divisible by } 3) = P(X = 3, 6, 9, 12) = 4/12 = 1/3.$

b) $P(X \text{ is divisible by } 5) = P(X = 5, 10) = 2/12 = 1/6.$

8A-3 Number of players = 24; number of players with even sizes = 11.
$\Rightarrow\ P(\text{even size}) = 11/24.\ \Rightarrow\ P(\text{odd size}) = 1 - 11/24 = 13/24.$

8A-4

a) $P(n \text{ even}) = P(n = 2, 4, 6, \ldots) = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots =$ geometric series with ratio $\frac{1}{4}$. $\Rightarrow$ probability $=\frac{1}{4}\cdot\frac{1}{1-1/4} = \frac{1}{3}.$

b) $P(n \text{ odd}) = 1 - P(n \text{ even}) = 2/3.$

c) We need to be slightly clever.
$P(X = 2 \text{ or } X = 3) = \frac{1}{4} + \frac{1}{8} = \frac{6}{16}, \qquad P(X \geq 5) = \frac{1}{2^5} + \frac{1}{2^6} + \ldots = \frac{1}{2^5}\cdot\frac{1}{1-1/2} = \frac{1}{16}.$
$\Rightarrow\ \frac{6}{16} = P(X = 2 \text{ or } X = 3) < P(X \text{ is prime}) < P(X = 2 \text{ or } X = 3 \text{ or } X \geq 5) = \frac{6}{16} + \frac{1}{16} = \frac{7}{16}.$ ■

8A-6 Let X = number of calls. X Poisson with mean 5 $\Rightarrow\ P(X = k) = \mathrm{e}^{-5}\cdot\frac{5^k}{k!}.$
$\Rightarrow\ P(X \geq 3) = 1 - P(X = 0 \text{ or } X = 1 \text{ or } X = 2) = 1 - \mathrm{e}^{-5}(1 + 5 + \frac{25}{2}) \approx .875.$

8A-7 Let X = number of errors on a page.
Sparse distribution $\Rightarrow$ Poisson of mean m $\Rightarrow\ P(X = k) = \mathrm{e}^{-m}\cdot\frac{m^k}{k!}.$
We're given $P(X = 0) = .2$. This $\Rightarrow\ \mathrm{e}^{-m} = .2\ \Rightarrow\ m = \ln(5) \approx 1.61.$
a) $P(\text{at most one error}) = P(X = 0 \text{ or } X = 1) = \mathrm{e}^{-m}(1 + m) \approx .52.$

b) Let X_3 = number of errors on 3 pages. This is still Poisson with mean = $3m$.
$\Rightarrow\ P(X_3 \leq 3) = P(X_3 = 0 \text{ or } X_3 = 1 \text{ or } X_3 = 2 \text{ or } X_3 = 3)$
$= \mathrm{e}^{-3m}(1 + (3m) + (3m)^2/2 + (3m)^3/6) \approx .29.$

8A-8 Let X = number of errors on a page.
Given $E(X) = \frac{600}{950} \approx .63 := m.$
Sparse $\Rightarrow$ Poisson with mean m $\Rightarrow\ P(X = k) = \mathrm{e}^{-m}\cdot\frac{m^k}{k!}.$
a) Let X_{10} = number of errors in 10 pages, X_{10} is Poisson with mean $10m$.
$\Rightarrow\ P(\text{no errors}) = P(X_{10} = 0) = \mathrm{e}^{-10m} \approx .002.$

b) Let X_5 = number or errors in 5 pages, X_5 is Poisson with mean $5m$.
$\Rightarrow\ P(\text{at least one error}) = P(X_5 \geq 1) = 1 - P(X_5 = 0) = 1 - \mathrm{e}^{-5m} \approx .96.$

(continued)

8B/2. Let t be the time between sales.
Exponential distribution $\Rightarrow\ f(t) = \frac{e^{-t/m}}{m}$. (Range of values $= [0,\infty)$.)
We're given mean $= m = 4/5$.

a) Using notes N1 and N4 above:

$$P(t > 2) = \int_2^\infty f(t)\,dt = -e^{-t/m}\Big|_2^\infty = -0 + e^{-2/m} = e^{-5/2} \approx \boxed{.082 = 8.2\%.}$$

b) $$P(t < 4) = \int_0^4 f(t)\,dt = -e^{-t/m}\Big|_0^4 = -e^{-4/m} + 1 = 1 - e^{-5} \approx = \boxed{.993 = 99\%.}$$

8B/3. Let t = waiting time between births.
Assume exponential $\Rightarrow\ f(t) = \frac{e^{-t/m}}{m}$. (Range of values $= [0,\infty)$.)
We're given mean $= m = 1/6$ minute.

$$P(1 \le t \le 2) = \int_1^2 f(t)\,dt = -e^{-t/m}\Big|_1^2 = -e^{-2/m} + e^{-1/m} = e^{-6} - e^{-12} \approx \boxed{.0025 = .25\%}$$

8B/5. Let t = waiting time between flats.
Assume exponential $\Rightarrow\ f(t) = \frac{e^{-t/m}}{m}$. (Range of values $= [0,\infty)$.)
We're given mean $= m = 100$.
Want time t_0 so that $P(t > t_0) = .9$.
Again using notes N1 and N4 above:

$$P(t > t_0) = \int_{t_0}^\infty f(t)\,dt = -e^{-t/m}\Big|_{t_0}^\infty = -0 + e^{-t_0/m} = e^{-t_0/100}.$$

Solving for t_0 so that $P(t > t_0) = .9$ gives $-t_0/100 = \ln(.9)$
$\Rightarrow\ t_0 = -100\ln(.9) \approx \boxed{10.536 \text{ days.}}$

8C/1a Mean $= m = 3.5 = 7/2$.
Method 1. $\sigma^2 = (\sum x_i^2 P(x_i)) - m^2$:

$$\sigma^2 = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) - (3.5)^2 = \frac{91}{6} - \frac{49}{4} = \frac{70}{24}. \ \Rightarrow\ \boxed{\sigma = \sqrt{70/24} \approx 1.708.}$$

Method 2. $\sigma^2 = \sum (x_i - m)^2 P(x_i)$:

$$\begin{aligned}\sigma^2 &= \frac{1}{6}((1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2) \\ &= \frac{1}{6}(\frac{25}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4}) = \frac{70}{24}.\end{aligned}$$

$\Rightarrow\ \boxed{\sigma = \sqrt{70/24} \approx 1.708.}$

8C/1b $f(x) = \dfrac{1}{x_2 - x_1}$.

$$m = E(X) = \int_{x_1}^{x_2} x \cdot \frac{1}{x_2 - x_1}\,dx = \frac{x_2^2 - x_1^2}{2(x_2 - x_1)} = \frac{x_2 + x_1}{2}.$$

$$\sigma^2 = \int_{x_1}^{x_2} x^2 f(x)\,dx - m^2 = \int_{x_1}^{x_2} x^2 \cdot \frac{1}{x_2 - x_1}\,dx - \left(\frac{x_2 + x_1}{2}\right)^2 = \text{(algebra)} = \frac{(x_2 - x_1)^2}{12}.$$

$\Rightarrow\ \boxed{\sigma = \dfrac{x_2 - x_1}{2\sqrt{3}}.}$

(continued)

8D/1b. From the table: $P(Z \leq -1) = 1 - P(Z \leq 1) = 1 - .8413 = \boxed{.1587 = 15.9\%.}$

8D/1d. From the table:
$P(Z < -1) = 1 - P(Z < 1) = 1 - .8413 = .1587, \quad P(Z < 2.5) = .9938.$
$\Rightarrow P(-1 < Z < 2.5) = .9938 - .1587 = \boxed{.8351 = 84\%}$

8D/2. Transform to a standard normal distribution:

$$\begin{aligned} P(85 < X < 135) &= P((85-m)/\sigma \\ &< (X-m)/\sigma < (135-m)/\sigma) \\ &= P(-35/36 < Z < 15/36) \\ &= P(-.97 < Z < .42) \\ &\approx P(-1 < Z < .4) \\ &= .6554 - (1 - .8413) \approx .5 \end{aligned}$$

<u>**answer:**</u> About 1/2 the 160 batteries will last between 85 and 135 hours.

8D/3. Let X be the normal random variable for the grades.

$$\begin{aligned} \Rightarrow P(\text{fail}) &= P(X < 55) = P(Z < (55-m)/\sigma) \\ &= P(Z < -15/10) = 1 - P(Z < 1.5) = 1 - .9332 = .0668. \end{aligned}$$

$\Rightarrow \boxed{\text{number of failures} = 300{*}.0668 = 20.}$

Let T be the passing threshold for the mean professor. $\Rightarrow .10 = P(X < T) = P(Z < (T-m)/\sigma)$.

From the table: $P(Z < 1.3) = .9 \Rightarrow P(Z < -1.3) = .1.$
$\Rightarrow (T-m)/\sigma = -1.3 \Rightarrow T = -1.3 \cdot 10 + 70 = \boxed{57.}$