18.02A Problem Set 5

(due Thurs., Nov. 1)

Part I (30 points)

TB = Simmons; SN = 18.02A Supplementary Notes (all have solutions)
The problems marked 'other' are not to be handed in.

Class 17 (Mon., Oct. 22) Vectors, dot product.
Read: TB: 17.3, 18.1, 18.2
Hand in: 1A/3a, 4a, 7bc, 8ab, 11; 1B/1a, 2a, 3a, 5b, 11, 13
Others: 1A/1, 2, 6

Class 18 (Tues., Oct. 23) Determinants, cross-product.
Read: SN: D, TB: 18.3
Hand in: 1C/2b, 3b, 4, 9; 1D/1b, 2, 5, 6
Others: 1C/1, 5; 1D/1a, 4

Class 19 (Wed., Oct. 24) Matrices, inverses.
Read: SN: M.1, M.2
Hand in: 1F/5b, 8a; 1G/3, 4, 5
Others: 1F/3

Continuation: (Thurs., Oct. 25) Discussion, review and catch up.

Class 20 (Mon., Oct. 29) Square matrices/systems, Cramer's rule, planes.
Read: SN: M.3, M.4
Hand in: 1H/3abc, 7; 1E/1cd, 2
Others: 1H/1, 5

Class 21 (Tues., Oct. 30) Parametric equations.
Read: TB: 17.1, 17.2 to middle of page 598, 18.4

Continuation: (Wed., Oct. 31) Discussion, review and catch up.

Class 22 (Thurs., Nov. 1 **pset 5 due**) Vector derivatives: velocity, curvature (2 hours).
Read: TB: 17.4, 17.5

(continued)

Part II (30 points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently.

Problem 1 (Class 17: 4 pts)
A man traveling east at a speed U finds that the wind seems to blow directly from the north. On doubling his speed he finds it appears to come from the northeast. Find the velocity (speed and direction) of the wind.

Problem 2 (Class 17: 4 pts)
Show that the lines joining the midpoints of the sides of a quadrilateral form a parallelogram.

Problem 3 (Class 18: 6 pts: 3,3)
a) Find the cosine of the angle between the main diagonal of a cube and one of the adjacent face diagonals.

b) Find the cosine of the angle between the two main diagonals.

Problem 4 (Class 18: 4 pts: 2,2)
The Biot-Savart law, which you will study in 8.02, says that the magnetic field $\mathbf{B}$ at a point (x, y, z) with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ induced by a current of magnitude J passing through the origin in the $\mathbf{k}$-direction is given by $\mathbf{B} = \dfrac{\mu_0}{4\pi} J \dfrac{\mathbf{k} \times \mathbf{r}}{r^3}$, where μ_0 is a physical constant and $r = |\mathbf{r}|$

a) Find the formula for the magnetic field $\mathbf{B}$ at a general point (x, y, z) and evaluate it at the point $(1, 2, 3)$.

b) For which points (x, y, z) at unit distance from the origin is the magnitude of the magnetic field, $|\mathbf{B}|$, largest?

Problem 5 (Class 19: 4 pts: 1,3 pts)
Consider the matrix $\begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ -4 & 4 & 2 \end{pmatrix}$.

a) Compute $\det A$.

b) Compute A^{-1} by computing in turn the following matrices:
i) minors, ii) cofactors, iii) adjoint; then dividing by $\det A$.
Check by matrix multiplication the the result found satisfies $AA^{-1} = I$.

Problem 6 (Class 20: 4 pts)
For what value of λ do the four points $(0, -1, -1)$, $(3, 9, 4)$, $(-4, 4, 4)$ and $(4, 5, \lambda)$ lie in a plane?

(continued)

Problem 7 (Class 20: 4 pts: 1,1,1,1) (Wordy, but not long)
This problem will introduce you to Matlab. This is a great tool, even if it is an expensive commercial package, which is widely used at MIT and in the engineering world. If you haven't already, you should get the document 'usingMatlab' off the course website.

This is not realistic but let's pretend that MIT students have trouble deciding on a major and every semester many of them change majors. To keep things simple we divide the majors into four categories: Math, Science, Engineering and Humanities.

Let the entries of the column vector $\mathbf{x} = [x_1, x_2, x_3, x_4]'$ represent respectively the fraction of students majoring in each of the four categories; for example, x_1 is the fraction of math majors. (Here, we use Matlab notation with the prime converting a row vector to a column vector)

Suppose after one semester 60% of math majors have stayed math majors, 30% of science majors have switch to math, 30% of engineers have switched to math and 40% of humanities majors have switched to math. Then if y_1, y_2, y_3, y_4 are the fractions after one semester, we can write

$$y_1 = .6x_1 + .3x_2 + .3x_3 + .4x_4$$

Assuming various rates of switching among majors after each semester we get a matrix equation

$$\mathbf{y} = A\mathbf{x}.$$

Changing names we let $\mathbf{x}^{(0)}$ be the initial fractions, $\mathbf{x}^{(1)}$ the fractions after 1 semester, $\mathbf{x}^{(2)}$ the fractions after 2 semesters etc. After carefully studying the statistics, the dean's office has determined that the switching matrix A has the values

$$\begin{pmatrix} .6 & .3 & .3 & .4 \\ .2 & .5 & .1 & .1 \\ .1 & .1 & .4 & .1 \\ .1 & .1 & .2 & .4 \end{pmatrix}$$

a) Assume the switching matrix A stays the same semester after semester, so that the vector $\mathbf{x}^{(2)} = A\mathbf{x}^{(1)}$, $\mathbf{x}^{(3)} = A\mathbf{x}^{(2)}$ etc. Suppose the initial fractions are (in percentages) 10, 40, 40, 10. Using Matlab, calculate $\mathbf{x}^{(n)}$ for $n = 1, 2, 5$ (use the operation raising matrices to powers)

b) Suppose college never ends. What will the fractions be after many semesters? In what semester do these 'final' fractions appear (round the numbers involved to 3 decimal places.)

c) Suppose the initial fractions (in percentages) were 5, 90, 5, 0. What are the final fractions in this case?

d) Explain why the columns of the matrix A all add up to 1.