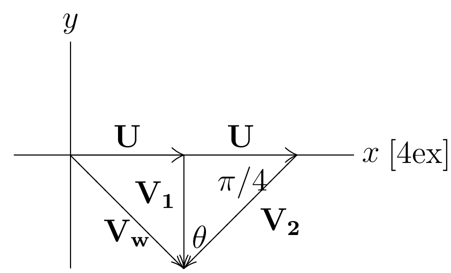
18.02A Problem Set 5 Solutions

Part II (30 points)

Note: Since we consider a point to be a vector (with tail at the origin), I'll often use the same notation, $\mathbf{A}$, for both. On occasion, I'll emphasize the vector nature by adding an arrow like so: $\overrightarrow{\mathbf{A}}$. The same statements apply to line segments: $\mathbf{AB}$ is both a segment and a vector, to emphasize the vector I may write $\overrightarrow{\mathbf{AB}}$.

Problem 1 (Class 17: 4 pts)
$\mathbf{V_w}$ = wind's velocity vector.
$\mathbf{U} = U\,\mathbf{i}$ = the man's original velocity vector
$\mathbf{V_1} = \mathbf{V_w} - \mathbf{U} = b\,\mathbf{j}$ = wind's original relative (apparent) velocity vector,
$2\mathbf{U} = \mathbf{U} + \mathbf{U}$ = new velocity vector
$\mathbf{V_2} = \mathbf{V_w} - 2\mathbf{U}$ = wind's new relative velocity vector.
(We assume that from the northeast means at 45°.)



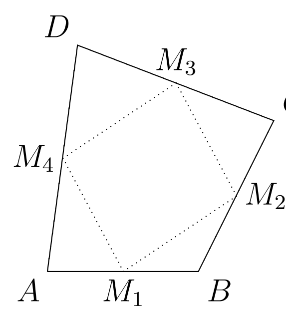
In the picture we see that angle $\theta = 45°$.

$\Rightarrow\ |\mathbf{V_1}| = U \ \Rightarrow\ \mathbf{V_w} = U\,\mathbf{i} - U\,\mathbf{j}$.

$\Rightarrow$ $\boxed{\text{the wind is from the northwest with speed } \sqrt{2}\,U.}$

Problem 2 (Class 17: 4 pts)
M_1, M_2, M_3, M_4 are the midpoints of the sides.

We need to show that $\overrightarrow{\mathbf{M_1M_2}} = \overrightarrow{\mathbf{M_4M_3}}$. (This will show two sides are parallel and the same length, the same argument then applies to the other pair of sides.)



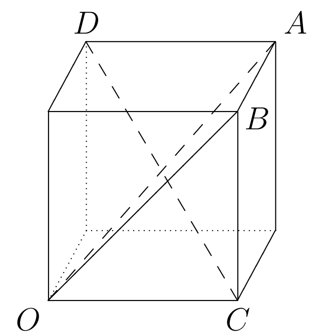
$\Leftrightarrow$ We must show $\overrightarrow{\mathbf{M_1M_2}} - \overrightarrow{\mathbf{M_4M_3}} = 0$. We have

$$\begin{aligned}
\overrightarrow{\mathbf{M_1M_2}} - \overrightarrow{\mathbf{M_4M_3}} &= \frac{1}{2}\overrightarrow{\mathbf{AB}} + \frac{1}{2}\overrightarrow{\mathbf{BC}} - \frac{1}{2}\overrightarrow{\mathbf{AD}} - \frac{1}{2}\overrightarrow{\mathbf{DC}} \\
&= \frac{1}{2}(\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}} + \overrightarrow{\mathbf{DA}} + \overrightarrow{\mathbf{CD}}) \\
&= \frac{1}{2}(\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}} + \overrightarrow{\mathbf{CD}} + \overrightarrow{\mathbf{DA}}) \\
&= 0. \ \blacksquare
\end{aligned}$$

(continued)

Problem 3 (Class 18: 6 pts: 3,3)
Put one corner of the cube at $O = (0,0,0)$ and the opposite corner at $A = (1,1,1)$ (so all the other vertices are at points with coordinates consisting of 0's and 1's. The vertex diagonally across the front face from O is $B = (1,0,1)$.

a) Let θ = angle between $\overrightarrow{\mathbf{OA}}$ and $\overrightarrow{\mathbf{OB}}$.
$\Rightarrow\ \overrightarrow{\mathbf{OA}}\cdot\overrightarrow{\mathbf{OB}} = |OA|\,|AB|\cos\theta = \sqrt{3}\,\sqrt{2}\cos\theta.$
Algebraically, $\overrightarrow{\mathbf{OA}}\cdot\overrightarrow{\mathbf{OB}} = \langle 1,1,1\rangle\cdot\langle 1,0,1\rangle = 2$
$\Rightarrow\ \boxed{\cos\theta = 2/\sqrt{6}.}$
$(\Rightarrow\ \theta = .615$ radians $= 35°.)$



b) We want the angle θ between $\overrightarrow{\mathbf{OA}}$ and $\overrightarrow{\mathbf{CD}}$.
$\overrightarrow{\mathbf{OA}} = \mathbf{i}+\mathbf{j}+\mathbf{k},\quad \overrightarrow{\mathbf{CD}} = \mathbf{j}+\mathbf{k}-\mathbf{i}.$
$\Rightarrow$ (geometrically) $\overrightarrow{\mathbf{OA}}\cdot\overrightarrow{\mathbf{CD}} = \sqrt{3}\cdot\sqrt{3}\cos\theta.$
(algebraically) $\overrightarrow{\mathbf{OA}}\cdot\overrightarrow{\mathbf{CD}} = 1.$
$\Rightarrow\ \boxed{\cos\theta = 1/3}\ \Rightarrow\ \theta = 71°.$

Problem 4 (Class 18: 4 pts: 2,2)
a) We could compute the cross product using determinants, but it's easier to compute directly: $\mathbf{k}\times\mathbf{r} = \mathbf{k}\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = x\,(\mathbf{k}\times\mathbf{i}) + y\,(\mathbf{k}\times\mathbf{j}) + z\,(\mathbf{k}\times\mathbf{k}) = x\mathbf{j} - y\mathbf{i}.$

$$\Rightarrow\ \boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\,J\,\frac{-y\mathbf{i}+x\mathbf{j}}{(x^2+y^2+z^2)^{3/2}}.}\qquad \boxed{B(1,2,3) = \frac{\mu_0}{4\pi}\,J\,\frac{-2\mathbf{i}+\mathbf{j}}{14^{3/2}}.}$$

b) $|\mathbf{B}| = \dfrac{\mu_0}{4\pi}\,J\,\dfrac{(x^2+y^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}}.$

Unit distance $\Rightarrow\ x^2+y^2+z^2 = 1$ and $x^2+y^2 = 1-z^2\ \Rightarrow\ |\mathbf{B}| = \dfrac{\mu_0}{4\pi}\,J\sqrt{1-z^2}.$

This is maximized when $z = 0\ \Rightarrow$ $\boxed{|\mathbf{B}| \text{ is largest at the points } (x,y,0) = (\cos\theta,\sin\theta,0).}$

(continued)

Problem 5 (Class 19: 4 pts: 1,3 pts)
a) $\det A = -2$

b) $A = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ -4 & 4 & 2 \end{pmatrix}$; minors $= \begin{pmatrix} -6 & 4 & -4 \\ 4 & -2 & 4 \\ 2 & -1 & 1 \end{pmatrix}$; cofactors $= \begin{pmatrix} -6 & -4 & -4 \\ -4 & -2 & -4 \\ 2 & 1 & 1 \end{pmatrix}$;

adjoint $= \begin{pmatrix} -6 & -4 & 2 \\ -4 & -2 & 1 \\ -4 & -4 & 1 \end{pmatrix}$; $\det A = -2$

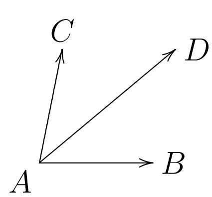
$$\Rightarrow A^{-1} = \tfrac{1}{-2}\begin{pmatrix} -6 & -4 & 2 \\ -4 & -2 & 1 \\ -4 & -4 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & -1/2 \\ 2 & 2 & -1/2 \end{pmatrix}.}$$

Problem 6 (Class 20: 4 pts)
Let $A = (0,-1,-1)$, $B = (3,9,4)$, $C = (-4,4,4)$, $D = (4,5,\lambda)$.
A, B, C, D are coplanar means $\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}$ is perpendicular to this plane and $\overrightarrow{\mathbf{AD}}$ is in the plane. $\Rightarrow$ $(\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}) \cdot \overrightarrow{\mathbf{AD}} = 0$.
$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{OB}} - \overrightarrow{\mathbf{OA}} = (3,10,5)$, likewise $\overrightarrow{\mathbf{AC}} = (-4,5,5)$ and $\overrightarrow{\mathbf{AD}} = (4,6,\lambda+1)$.

$$\Rightarrow \overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} = 25\mathbf{i} - 35\mathbf{j} + 55\mathbf{k}.$$

Therefore, $(\overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}}) \cdot \overrightarrow{\mathbf{AD}} = 0$
$\Leftrightarrow$ $(25,-35,55)\cdot(4,6,\lambda+1) = 100 - 210 + (\lambda+1)\,55 = 0 \;\Leftrightarrow\; \lambda = 1$.

$\boxed{\text{They are coplanar when } \lambda = 1.}$



Problem 7 (Class 20: 4 pts: 1,1,1,1) (Wordy, but not long)
a) We use Matlab notation to write column vectors as a primed row vector.
$\mathbf{x}^{(0)} = [.1, .4, .4, .1]'$, $\mathbf{x}^{(1)} = A\mathbf{x}^{(0)} = [.34, .27, .22, .17]'$,
$\mathbf{x}^{(2)} = A\mathbf{x}^{(0)} = [.419, .242, .166, .173]'$, $\mathbf{x}^{(5)} = A\mathbf{x}^{(0)} = [.452, .241, .144, .164]'$.

b) Longterm $= \mathbf{x}^{(n)}$ for n large $= [.452, .242, .143, .163]'$: First occurs when $n = 7$.

c) Same is in part (b).

d) Everybody has to go somewhere. For example, the first column represents the fraction of math majors who stay math majors or switch to science, engineering or humanities. The total has to be all current math majors, i.e. 100 percent.