18.02A Problem Set 6 Solutions

Part II (29 points)

Problem 1 (Class 21, 5 pts: 1,1,2,1)

a) The paremeter t is in units of time.

Usual unit circle is $(x, y) = (\cos t, \sin t)$ (counterclockwise at unit speed, starting at $(1, 0)$).

Reverse direction: $(x, y) = (\cos(-t), \sin(-t)) = (\cos t, -\sin t)$.

Shift start to $(-1, 0)$: $(x, y) = (\cos(t + \pi), -\sin(t + \pi)) = (-\cos t, \sin t)$.

$\Rightarrow$ position vector $= \boxed{\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}.}$

b) Circular CCW at constant speed $\Rightarrow$ $(x, y) = 10(\cos \omega t, \sin \omega t)$.

Speed $= \sqrt{(x')^2 + (y')^2} = 10\omega = 60 \Rightarrow \omega = 6. \Rightarrow \boxed{\mathbf{r}(t) = 10\cos(6t)\,\mathbf{i} + 10\cos(6t)\,\mathbf{j}.}$

c) RPM is revolutions (or cycles) per minute.

60 rpm $\Leftrightarrow$ 120π radians/minute $\Rightarrow$ $\boxed{\mathbf{r}(t) = 10\cos(120\pi\, t)\,\mathbf{i} + 10\sin(120\pi\, t)\,\mathbf{j}).}$

d) Because they are so easy, we don't show all the algebra for the integrals. The important thing is to remember the constant of integration.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\,\mathbf{k} + \mathbf{c_1}; \quad \mathbf{v_0} = -\mathbf{i} \Rightarrow \mathbf{c_1} = \langle -1, -1, 0\rangle.$$

$$\Rightarrow \mathbf{v} = (-1 + \sin t)\,\mathbf{i} + (-1 + \cos t)\,\mathbf{j} + t\,\mathbf{k}.$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = (-t - \cos t)\,\mathbf{i} + (-t + \sin t)\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k} + \mathbf{c_2}; \quad \mathbf{r_0} = \mathbf{j} \Rightarrow \mathbf{c_2} = \langle 1, 1, 0\rangle.$$

$$\Rightarrow \boxed{\mathbf{r}(t) = (1 - t - \cos t)\,\mathbf{i} + (1 - t + \sin t)\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k}.}$$

Problem 2 (Class 21, 8 pts: 2,3,2,1 + 2 E.C.)

a) Let P be the point giving the jet's position. So the position vector is

$\overrightarrow{\mathbf{OP}} = \mathbf{r}(t) = \langle 1, 1, 0\rangle + t\,\langle -5, 0, 1\rangle = (1 - 5t)\,\mathbf{i} + \mathbf{j} + t\,\mathbf{k}$.

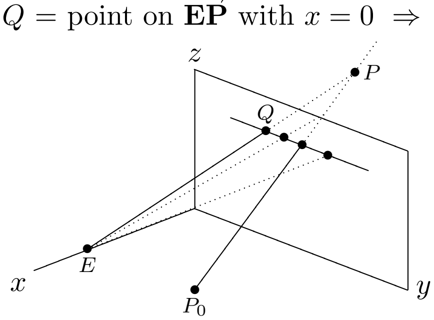
We want to know where on the yz-plane the eye at point E will see the plane. Call the point on the yz-plane Q.

$\Rightarrow$ Q = intersection of the line $\overrightarrow{\mathbf{EP}}$ with the yz-plane.

Line $\overrightarrow{\mathbf{EP}}$ is parameterized by (need to use u because t is already taken)

$$(x, y, z) = E + u(P - E) = (1, 0, 0) + u(-5t, 1, t).$$

Q = point on $\overrightarrow{\mathbf{EP}}$ with $x = 0 \Rightarrow 1 - 5u\,t = 0 \Rightarrow u = 1/5t$.



$\Rightarrow$ $\boxed{Q = (0, 1/5t, 1/5) \quad \text{or} \quad y = 1/5t,\ z = 1/5.}$

(continued)

b) We simply repeat the answer to part (a) using letters instead of numbers.

Position of jet: $\overrightarrow{\mathbf{OP}} = \mathbf{r}(t) = \langle a + \alpha t,\, b + \beta t,\, c + \gamma t\rangle$.

To the eye at $E = (1, 0, 0)$ the jet at point P will appear on the screen at the point Q where $\overrightarrow{\mathbf{EP}}$ intersects the yz-plane.

$\overrightarrow{\mathbf{EP}}$ is parametrized by $(x, y, z) = E + u\,(P - E) = (1, 0, 0) + u\,(a - 1 + \alpha t,\, b + \beta t,\, c + \gamma t)$.

The point Q is the point on $\overrightarrow{\mathbf{EP}}$ with $x = 0 \;\Rightarrow\; 1 + u(a - 1 + \alpha t) = 0 \;\Rightarrow\; u = \dfrac{1}{1 - a - \alpha t}$

$$\Rightarrow \boxed{y = \frac{b + \beta t}{1 - a - \alpha t}, \quad z = \frac{c + \gamma t}{1 - a - \alpha t}.}$$

c)

You weren't asked to do this, but we show the trajectory in part (b) is a straight line. For this, we compute velocity. After the quotient rule and some algebra we have

$$(y', z') = \left(\frac{\beta(1 - a) + b\alpha}{(1 - a - \alpha t)^2}, \frac{\gamma(1 - a) + c\alpha}{(1 - a - \alpha t)^2}\right)$$
$$= \frac{1}{(1 - a - \alpha t)^2}\left(\beta(1 - a) + b\alpha,\, \gamma(1 - a) + c\alpha\right).$$

$\Rightarrow$ the velocity always points in same direction (along $(\beta(1 - a) + b\alpha,\, \gamma(1 - a) + c\alpha)$) $\Rightarrow$ the trajectory on screen is along a line.

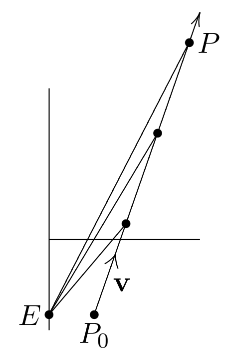


Figure 1

(2) From the formula for (y, z) in part (b) we have $\boxed{\lim_{t\to\infty} (y, z) = (-\beta/\alpha, -\gamma/\alpha).}$

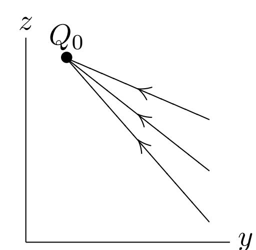


Figure 2

d) See figure 2

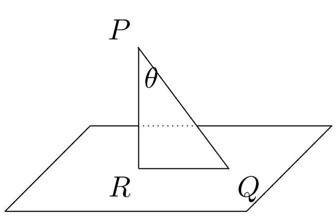
Extra credit) The (two dimensional) figure 1 shows that as $t \to \infty$ the line $\overrightarrow{\mathbf{EP}}$ becomes parallel to $\mathbf{v}$, i.e. it heads towards the line $E + u\,\mathbf{v}$.

$\Rightarrow$ the image on the screen heads towards the point Q_0 where the line $E + u\,\mathbf{v}$ intersects the yz-plane.

This line is $(x, y, z) = (1, 0, 0) + u(\alpha, \beta, \gamma) \;\Rightarrow\;$ it intersects the yz-plane when $u = -1/\alpha \;\Rightarrow\; y = -\beta/\alpha, \quad z = -\gamma/\alpha$ (as in part (c)).

Problem 3 (Class 21, 4 pts: 2,2)

a)



Q = any point on the plane, we take $Q = (7, 0, 0)$.

$\mathbf{N}$ = normal to plane $= \langle 1, 1, 0\rangle = \mathbf{i} + \mathbf{j}$.

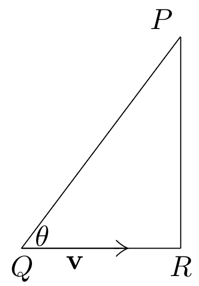
R = point on plane closest to P

Distance $= |PR| = |PQ|\cos\theta = \left|\overrightarrow{\mathbf{PQ}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|}\right|$

$\overrightarrow{\mathbf{PQ}} = \langle 6, -1, -1\rangle, \quad |\mathbf{N}| = \sqrt{2} \;\Rightarrow\; \overrightarrow{\mathbf{PQ}} \cdot \dfrac{\mathbf{N}}{|\mathbf{N}|} = \boxed{\dfrac{5}{\sqrt{2}}.}$

(continued)

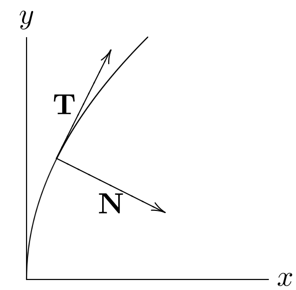
b) We do this using the cross product method. Projection works just as well.



Q = any point on the line, we take $Q = (1, 2, 2)$.
$\mathbf{v}$ = direction vector of line $= (1, 2, 2)$.
$\overrightarrow{\mathbf{QP}} = \langle 0, -1, -1\rangle$.
R = point on line closest to P: we want $|PR|$.
$|PR| = |QP|\sin\theta = |\overrightarrow{\mathbf{QP}} \times \mathbf{v}|$

$$\overrightarrow{\mathbf{QP}} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = -\mathbf{j} + \mathbf{k} \Rightarrow \boxed{|PR| = \sqrt{2}/3.}$$

Problem 4 (Class 22: 5 pts: 3,1,1)
a)



$\mathbf{r}(t) = at^2\,\mathbf{i} + 2at\,\mathbf{j}$
$\mathbf{v}(t) = \mathbf{r}' = 2at\,\mathbf{i} + 2a\,\mathbf{j}$
$\mathbf{a}(t) = \mathbf{r}'' = 2a\,\mathbf{i}$

$$\boxed{\text{Unit tangent} = \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t}{\sqrt{1+t^2}}\,\mathbf{i} + \frac{1}{\sqrt{1+t^2}}\,\mathbf{j}.}$$

We use the formulas:
1) $\kappa = \dfrac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3}$.
2) $\mathbf{v}\times(\mathbf{a}\times\mathbf{v}) = \kappa\,|v|^4\,\mathbf{N}$. ($\Rightarrow$ $\mathbf{N}$ is the unit vector parallel to $\mathbf{v}\times(\mathbf{a}\times\mathbf{v})$.)
Computing: $|v| = 2a\sqrt{1+t^2}$, $\mathbf{a}\times\mathbf{v} = 4a^2\,\mathbf{k}$, $\mathbf{v}\times(\mathbf{a}\times\mathbf{v}) = 8a^3\,\mathbf{i} - 8a^3t\,\mathbf{j}$.
$\mathbf{N}$ = unit vector in direction of $\mathbf{v}\times(\mathbf{a}\times\mathbf{v})$ =

$$\Rightarrow \boxed{\mathbf{N} = \frac{1}{\sqrt{1+t^2}}\,\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\,\mathbf{j}.} \quad \text{(It's easy to check } \mathbf{N}\perp\mathbf{T}.)$$

Center of curvature C: $\overrightarrow{\mathbf{OC}} = \mathbf{r} + R\mathbf{N} = \langle at^2,\ 2at\rangle + 2a(1+t^2)\langle 1, -t\rangle \Rightarrow \boxed{C = (2a + 3at^3,\ -2at^3).}$

b) Since y is a function of x the following parametrization: $x = t$, $y = 2t + 3$.
$\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{a} = \mathbf{0} \Rightarrow \kappa = 0 \Rightarrow \boxed{\text{Rad. of curvature} = \infty.}$
(All this should have been expected since the curve is a line.)

c) Parametrization: $x = t$, $y = t^2 \Leftrightarrow \mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j}$.
$\Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j}$, $\mathbf{a} = 2\mathbf{j}$.
$\Rightarrow \mathbf{a}\times\mathbf{v} = -2\mathbf{k} \Rightarrow \kappa = \dfrac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \dfrac{2}{(1+4t^2)^{3/2}}$.
By inspection κ is maximized when $t = 0$, i.e. the vertex $(0, 0)$ is the point of maximum curvature.

(continued)

Problem 5 (Class 20: 3 pts)
To have names, let $A = (1, 0, 0)$, $B = (0, 2, 0)$ and $C = (0, 0, 1)$.
The center is at the intersection of three planes:
1. the plane that perpendicularly bisects AB
2. the plane that perpendicularly bisects AC
3. the plane containing the 3 points A, B, and C.

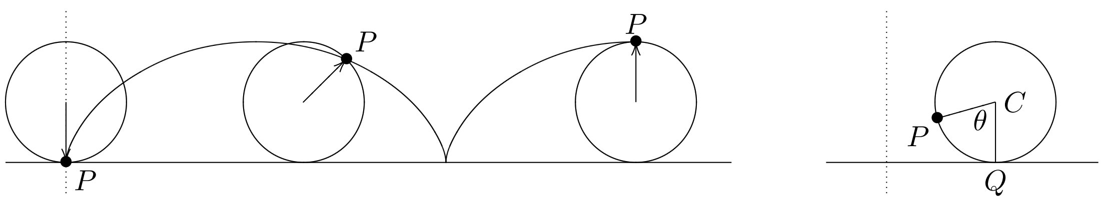
To get the equation for each plane we need a normal $\mathbf{N}$ and a point P:
1. $\mathbf{N} = \overrightarrow{\mathbf{AB}} = \langle -1, 2, 0\rangle$, $P = \frac{A+B}{2} = (1/2, 1, 0)$: $\Rightarrow\ -x + 2y = 3/2$.
2. $\mathbf{N} = \overrightarrow{\mathbf{AC}} = \langle -1, 0, 1\rangle$, $P = \frac{A+C}{2} = (1/2, 0, 1/2)$: $\Rightarrow\ -x + z = 0$.
3. $\mathbf{N} = \overrightarrow{\mathbf{AB}} \times \overrightarrow{\mathbf{AC}} = \langle 2, 1, 2\rangle$, $P = (1, 0, 0)$: $\Rightarrow\ 2x + y + 2z = 2$.

Intersection of 3 planes $\Leftrightarrow$ solve 3 equations in 3 unknowns:

$$-x + 2y = 3/2; \quad -x + z = 0; \quad 2x + y + 2z = 2.$$

These can be solved my matrix methods or elimination (in this case I think elimination is easier) to get: **answer:** Center = $(5/18, 8/9, 5/18)$.

Problem 6 (Classes 22, 4 pts: 2,2)
a) Definition: the cycloid is the trajectory of a point P on a wheel of radius a as it rolls along the x-axis.



We parametrize using θ = the angle through which the wheel has turned.
Assume the point P starts at the origin):
$\overrightarrow{\mathbf{OP}} = \mathbf{r}(\theta) = \overrightarrow{\mathbf{OQ}} + \overrightarrow{\mathbf{QC}} + \overrightarrow{\mathbf{CP}}$.
$\overrightarrow{\mathbf{OQ}} = a\theta\,\mathbf{i}$, ($a\theta$ = distance rolled).
$\overrightarrow{\mathbf{QC}} = a\,\mathbf{j}$
$\overrightarrow{\mathbf{CP}} = -a\sin\theta\,\mathbf{i} - a\cos\theta\,\mathbf{j}$.
$\Rightarrow\ \mathbf{r}(\theta) = a(\theta - \sin\theta)\,\mathbf{i} + a(1 - \cos\theta)\,\mathbf{j}$. $\Rightarrow$ $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$.
NOTE: the symmetric form of equations is hard to write down

b) Differential of arclength

$$\begin{aligned} ds &= \sqrt{(x')^2 + (y')^2}\ d\theta = \sqrt{(a - a\cos\theta)^2 + (a\sin\theta)^2}\ d\theta \\ &= a\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\ &= a\sqrt{2(1-\cos\theta)}\, d\theta = 2a\sqrt{\tfrac{1-\cos\theta}{2}}\, d\theta = 2a|\sin\tfrac{\theta}{2}|\, d\theta. \end{aligned}$$

$$\begin{aligned} \text{Arclength of arch} &= \int_0^{2\pi} 2a|\sin\tfrac{\theta}{2}|\, d\theta \\ &= \int_0^{2\pi} 2a\sin\tfrac{\theta}{2}\, d\theta \quad (\text{ since } \sin\theta/2 \geq 0 \text{ for } 0 < \theta < 2\pi) \\ &= -4a\cos\tfrac{\theta}{2}\Big|_0^{2\pi} = 8a. \end{aligned}$$