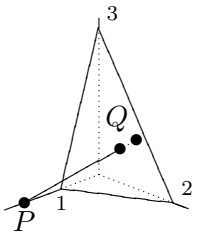
18.02A Problem Set 6 Worked Examples

Problem 1 (Class 21, 2 pts)
Let F be semi-transparent plane containing the points $x = 1$, $y = 2$ and $z = 3$ on the three axes respectively. If you put your eye at the point $(2, 0, 0)$ at what point on F will you see think you see a point of light at $(0, 1, 1)$?

answer:



Let $P = (2, 0, 0)$ and $Q = (0, 1, 1)$.
We want the intersection of the line PQ and the plane F.
Line: $(2, 0, 0) + t(-2, 1, 1) = (2 - 2t, t, t)$.
Plane: $x + \frac{y}{2} + \frac{z}{3} = 1$.
Intersection: $(2 - 2t) + \frac{t}{2} + \frac{t}{3} = 1 \;\Rightarrow\; t = \frac{6}{7}$.
$\Rightarrow$ pt. of intersection $= (2/7, 6/7, 6/7)$.

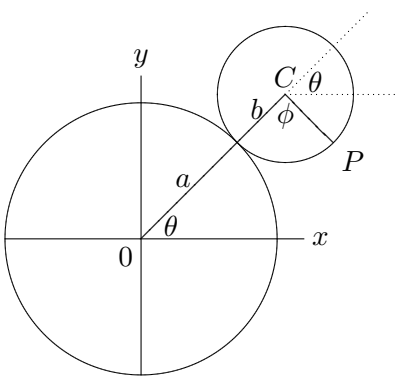
Problem 2 (Class 21: 3 pts)
The hypocycloid is discussed on page 594-595 of the text. This is the curve formed by following a point on a circle as it rolls around the inside of another circle. Similarly, an epicycloid is traced out by a point on a circle rolling around the outside of another circle.
Assume:
-the fixed circle has radius a and is centered at the origin.
-the rolling circle has radius b.
-the point tracing out the curve is called P
-the rolling circle starts with its center on the positive x-axis with the point P at $(a, 0)$.
Using the hypocycloid as an example give parametric equations for the epicycloid.

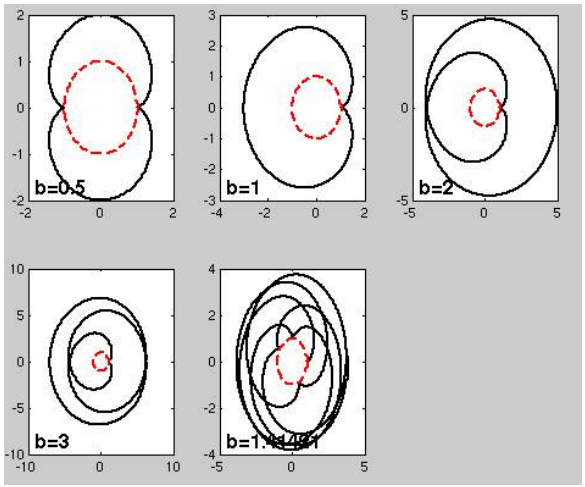
answer:
Equating arclength along each circle we get: $a\theta = b\phi$.
$\overrightarrow{\mathbf{OC}} = (a + b)(\cos\theta\,\widehat{\mathbf{i}} + \sin\theta\,\widehat{\mathbf{j}})$.

$$\begin{aligned}\overrightarrow{\mathbf{CP}} &= b(\cos(\pi - (\theta + \phi))\,\widehat{\mathbf{i}} - \sin(\pi - (\theta + \phi))\,\widehat{\mathbf{j}}) \\ &= b(-\cos(\theta + \phi)\,\widehat{\mathbf{i}} - \sin(\theta + \phi)\,\widehat{\mathbf{j}}).\end{aligned}$$

$$\begin{aligned}\mathbf{r}(\theta) = \overrightarrow{\mathbf{OP}} &= \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{CP}} \\ &= [(a + b)\cos\theta - b\cos(\theta + \phi)]\,\widehat{\mathbf{i}} + [(a + b)\sin\theta - b\sin(\theta + \phi)]\,\widehat{\mathbf{j}} \\ &= [(a + b)\cos\theta - b\cos((1 + a/b)\theta)]\,\widehat{\mathbf{i}} + [(a + b)\sin\theta - b\sin((1 + a/b)\theta)]\,\widehat{\mathbf{j}}\end{aligned}$$

$\Rightarrow\; x(\theta) = (a + b)\cos\theta - b\cos((1 + a/b)\theta),\; y(\theta) = (a + b)\sin\theta - b\sin((1 + a/b)\theta)$





These plots were made with Matlab using $a = 1$ and various values of b.