18.02a Problem Set 7

(due Thurs., Nov. 29)

Part I (20 points)

TB = textbook; SN = Supplementary Notes (all have solutions)

Class 26 (Thurs., Nov. 15) Tangent plane approximation, directional derivatives.
Read: TB: 19.1, 19.2 SN: TA
Hand in: 2D/1abc, 2a, 3a, 4, 5ab, 7, 9bcde
Others: 2D/1de, 2b

Class 27 (Mon., Nov. 19) Chain rule.
Read: TB: 19.6
Hand in: 2E/1b, 2c, 3a, 4, 7, 8a
Others: 2E/1, 2, 5

Class 28 (Tues., Nov. 20) Max-min problems, least squares.
Read: TB: 19.7, SN: LS
Hand in: 2F/1b, 5; 2G/1c, 4
Others: 2F/1a, 2; 2G/1a, 4, 5

No class Wed. Nov. 21.

Class 29 (Mon., Nov. 26) Second derivative test, Lagrange multipliers.
Read: TB: 19.7, 19.8
Hand in: 2H/1ad; 2I/1a, 3
Others: 2H/1bc, 5; 2I/1b, 2

Class 30 (Tues., Nov 27) Non-independent variables, chain rule.
Read: TB: 19.6, SN: N.1-N.3

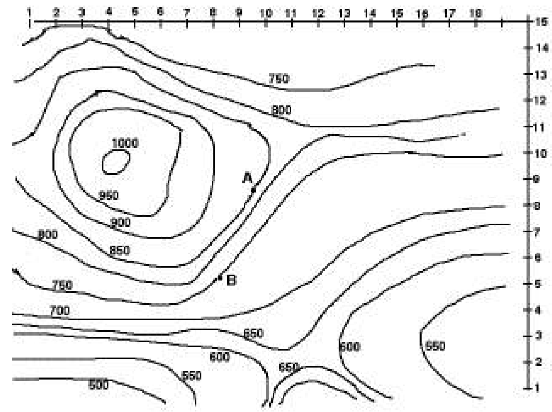
Part II (33 points)

Problem 1 (Class 26, 4 pts)
Consider the surface $xyz = d$ in the first octant and a point $P_o = (x_o, y_o, z_o)$ on the surface. The tangent plane to the surface at the point P_o intersects each of the positive axes. These intercepts and the origin are the vertices of a tetrahedron. Show this tetrahedron has constant (i.e. independent of (x_o, y_o, z_o)) volume. (The volume of a tetrahedron is $\frac{1}{3}bh$ where b is the base and h is the height.).

Hint: first do the analogous problem for $xy = d$ in the plane and the triangle created by the tangent line and the axes. When doing this use the point-normal form to write the equation of the tangent line.

(continued)

Problem 2 (Class 26, 5 pts:1,1,1,1,1)
The picture shows the level curves for a height function $h(x, y)$. The scale is given along the right and top of the picture.



a) Estimate h_x and h_y at point A.

b) Estimate $\frac{dh}{ds}$ at the point B in the direction of $-\mathbf{i} + \mathbf{j}$.

c) Find the point P with the largest x coordinates such that $h(P) = 900$ and $h_x(P) = 0$. Find the point Q with the largest x coordinates such that $h(Q) = 950$ and $h_y(Q) = 0$.

d) Find the point R on the level curve $h = 900$ with the smallest x coordinates such that $\frac{dh}{ds} = 0$ in the direction of $\mathbf{i} + \mathbf{j}$.

e) Find the point S such that $\nabla h = 0$ but S is not a minimum or a maximum.

Problem 3 (Class 26, 5 pts:2,2,1)
A hiker climbs a mountain whose height is given by $z = 1000 - 2x^2 - 3y^2$.

a) When the hiker is at the point $(1, 1, 995)$, in what direction (on her 2D topographical map) should she move in order to ascend as rapidly as possible?

b) The answer in part (a) corresponds to what direction along the mountain, i.e., in 3D space?

c) If she continues on a path of steepest ascent, show that the projection of this path on the xy-plane (i.e. the path on her topo. map) is $y = x^{3/2}$.

Problem 4 (Class 29, 5 pts:1,2,2)
Place a unit cube in the corner of the first octant with edges along the axes. For this problem consider the front face diagonal containing $(1, 0, 1)$ and the right face diagonal containing $(0, 1, 0)$. These two lines are skew; the problem is to find the length and position of the shortest line segment joining them.

a) Draw a picture and write parametric equations for the two lines containing these two diagonals. For clarity, use different variables, t and u, as the parameters for the two lines.

b) Let $w(t, u)$ be the *square* of the distance between a point on A on the front diagonal and a point B on the side diagonal. Find the (unique) critical point for the function $w(t, u)$ and use this to find the positions and length of the shortest line segment between the two lines.

c) Use the second derivative test to verify that the length of this line segment is a minimum.

(continued)

Problem 5 (Class 28, 7 pts:2,2,2,1)
a) Let $\mathbf{x} = [x_1 \ldots x_n]$, $\quad \mathbf{y} = [y_1 \ldots y_n]$, $\quad \mathbf{i} = [1 \ldots 1]$ (n 1's).
Let $Y = aX + b$ be the best fitting line to the n points (x_i, y_i).

Translate equations (4) in the notes §LS into a 2×2 matrix equation $A \begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{d}$.

Write the entries of A and $\mathbf{d}$ in terms of the vector operations on $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{i}$
(e.g. write $\sum x_i$ as $\mathbf{x} \cdot \mathbf{i}$).

b) The cost of a first class stamp over the last 44 years is given in the following table.

.05 (1963)	.06 (1968)	.08 (1971)	.10 (1974)	.13 (1975)	.15 (1978)
.20 (1981)	.22 (1985)	.25 (1988)	.29 (1991)	.32 (1995)	.33 (1999)
.34 (2001)	.37 (2002)	.39 (2006)	.41 (2007)		

Use Matlab to produce a scatter plot of this data. For consistancy everyone should use units of time in years since 1960 and prices in cents.
(The command: plot(x,y,'*') will put a * at each data point.)

c) In Matlab make the matrix A and the vector $\mathbf{d}$ and solve for a and b. Superimpose the line $at + b$ on the scatter plot. (Use the 'hold' command then let $\mathbf{t} = [0\ 50]$ and plot $a\mathbf{t} + b$.)

d) Estimate the cost of a stamp in 2010.

Problem 6 (Class 29, 3 pts)
Consider all the planes passing through the point $(4, 2, 1)$. Using Lagrange multipliers find the plane that cuts off the tetrahedron in the first octant having the smallest volume.
Hint: for symmetry use as variables the intercepts a, b and c.

Problem 7 (Class 26, 4 pts:1,1,2)
a) For a square of width x and height y give the tangent plane approximation for the area when $x = 5$ and $y = 2$

b) Use the approximation formula to estimate the area when $x = 2.1$ and $y = 5.2$

c) Draw the rectangle in part (b) and inside it the one from part (a). Indicate on the drawing Δx and Δy. Use the drawing to explain both the approximation formula and the error in the approximation geometrically.