18.02a Problem Set 7 Solutions

Part II (33 points)

Problem 1 (Class 26, 4 pts)

Let $w = xyz$, our surface = level surface: $w = d$.

Normal to surface = $\nabla w = (yz, xz, xy)$.

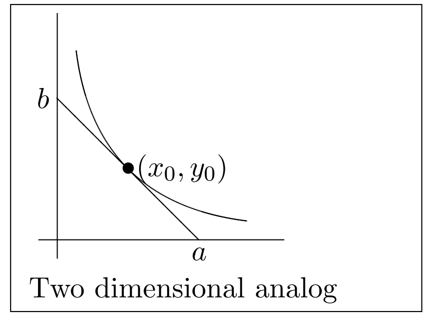
Evaluate at $P_o \Rightarrow$ normal = $(y_o z_o, x_o z_o, x_o y_o)$.

Tangent plane (point-normal form):

$y_o z_o(x - x_o) + x_o z_o(y - y_o) + x_o y_o(z - z_o) = 0$.

Rewriting tangent plane: $y_o z_o \cdot x + x_o z_o \cdot y + x_o y_o \cdot z = 3x_o y_o z_o = 3d$.

x, y, z intercepts: $a = \dfrac{3d}{y_o z_o}, \quad b = \dfrac{3d}{x_o z_o}, \quad c = \dfrac{3d}{x_o y_o}$.



Two dimensional analog

$$\Rightarrow \text{ volume } = \frac{1}{6}abc = \frac{27d^3}{6x_o^2 y_o^2 z_o^2} = \frac{27d^3}{6d^2} = \frac{9d}{2} \quad \text{(constant independent of } P_o\text{)}.$$

Problem 2 (Class 26, 5 pts:1,1,1,1,1)

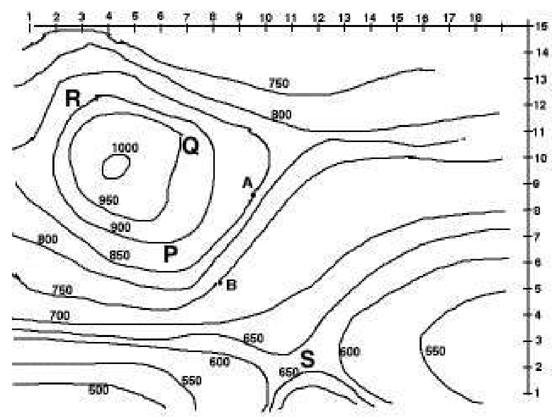
a) $h_x \approx \frac{-50}{.5} = -100$ (measuring from curve $h = 850$ to $h = 800$) or $h_x \approx \frac{-100}{2} = -50$ (from $h = 900$ to $h = 800$.)
$h_y \approx \frac{-50}{-.5} = 100$ (from $h = 850$ to $h = 800$).

b) $\frac{dh}{ds} \approx \frac{50}{1} = 50$ (from $h = 750$ to $h = 800$)

c) $P = (5.75, 6.75)$ (horizontal tangent).
$Q = (6.75, 11)$ (vertical tangent)

d) $R = (3, 12)$ (tangent at 45°)

e) $S = (11, 2)$ (mountain pass or saddle)



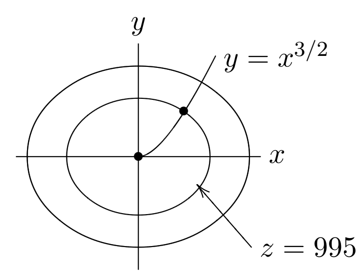
Problem 3 (Class 26, 5 pts:2,2,1)

(a) The direction in the xy-plane of maximum increase of z is in the direction of $\nabla z = \langle -4x, -6y \rangle \Rightarrow \nabla z|_{(1,1)} = \langle -4, -6 \rangle$.

$\Rightarrow$ (in the xy-plane, the fastest increase is in direction of $\langle -4, -6 \rangle$.)

(b) If the projection of her path in the xy-plane has tangent vector $a\,\mathbf{i} + b\,\mathbf{j}$ then the tangent vector of her path on the mountain is $a\,\mathbf{i} + b\,\mathbf{j} + (\nabla z \cdot \langle a, b \rangle)\,\mathbf{k}$.

$\Rightarrow$ along the mountain the fastest increase is in direction of $-4\,\mathbf{i} - 6\,\mathbf{j} + 52\,\mathbf{k}$.



Level curves

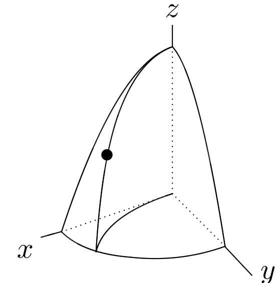
c) We need to show that the tangent at each point along the graph of $y = x^{3/2}$ is parallel to ∇z.

Point on graph: $P = (x, x^{3/2})$.

Tangent vector to $y = x^{3/2}$ at P is $\mathbf{v} = \langle 1, \frac{3}{2}x^{1/2} \rangle$.

$\nabla z|_P = \langle -4x, -6x^{3/2} \rangle = -4x\,\mathbf{v}$. So they are parallel . QED

(The 3d picture at right shows the path along the mountain.)

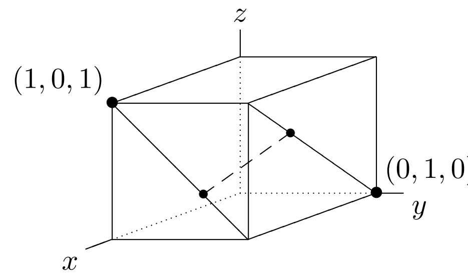


(continued)

Problem 4 (Class 29, 5 pts:1,2,2)

a) Line 1: $(1,0,1)+t(0,1,-1) \Rightarrow \boxed{x=1, \quad y=t, \quad z=1-t.}$

Line 2: $(0,1,0)+u(1,0,1) \Rightarrow \boxed{x=u, \quad y=1, \quad z=u.}$



b) $A=(1,t,1-t), \quad B=(u,1,u)$

$\Rightarrow w(t,u)=|AB|^2=(u-1)^2+(1-t)^2+(u+t-1)^2.$

$$\Rightarrow \frac{\partial w}{\partial t} = -2(1-t)+2(u+t-1)=4t+2u-4,$$

$$\frac{\partial w}{\partial u} = 2(u-1)+2(u+t-1)=4u+2t-4.$$

Critical point when $\frac{\partial w}{\partial t}=\frac{\partial w}{\partial u}=0 \Rightarrow \begin{cases} 4t+2u-4 &= 0 \\ 2t+4u-4 &= 0. \end{cases}$

Solving by any method you like $\Rightarrow u=2/3, \ t=2/3.$

$\Rightarrow \boxed{A_0=(1,\frac{2}{3},\frac{1}{3}), \quad B_0=(\frac{2}{3},1,\frac{2}{3}), \quad |A_0B_0|=\frac{1}{\sqrt{3}}.}$

c) $A=\frac{\partial^2 w}{\partial^2 t}=4, \quad C=\frac{\partial^2 w}{\partial^2 u}=4, \quad B=\frac{\partial^2 w}{\partial t \partial u}=2.$

Second derivative test: $AC-B^2=12>0$ and $A>0 \Rightarrow$ minimum.

Problem 5 (Class 28, 7 pts:2,2,2,1)

a) $A=\begin{pmatrix} \mathbf{x}\cdot\mathbf{x} & \mathbf{x}\cdot\mathbf{i} \\ \mathbf{x}\cdot\mathbf{i} & \mathbf{i}\cdot\mathbf{i} \end{pmatrix}, \quad \mathbf{d}=\begin{pmatrix} \mathbf{x}\cdot\mathbf{y} \\ \mathbf{i}\cdot\mathbf{y} \end{pmatrix}.$

b) See below.

c) Using matlab:

$$\begin{pmatrix} 14266 & 424 \\ 424 & 16 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 12401 \\ 369 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} .86551 \\ .12644 \end{pmatrix}$$

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%Matlab code
x = [3 8 11 14 15 18 21 25 28 31 35 39 41 42 46 47];
y = [5 6 8 10 13 15 20 22 25 29 32 33 34 37 39 41];
i = ones(1, length(x));
A = [x*x' x*i'; x*i' i*i']
d = [x*y' i*y']'
u = inv(A)*d;
a = u(1)
b = u(2)
clf %clear figure
plot(x,y,'*k') %scatter plot
hold on
t = [0 50];
plot(t, a*t + b, 'k') %line
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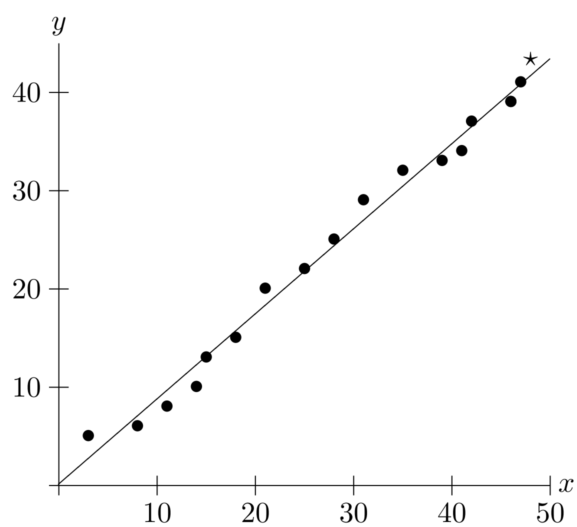
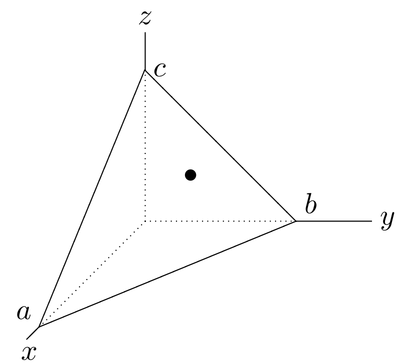


d) $t=50 \Rightarrow$ predicted cost $= a*50+b=43.4$. **answer:** \$.43.

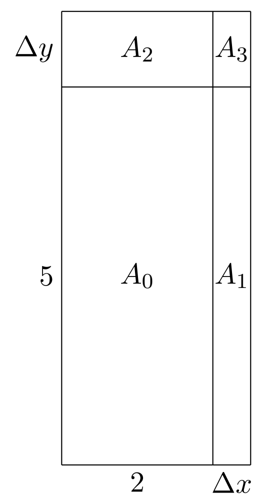
(continued)

Problem 6 (Class 29, 3 pts)



Equation of plane: $\frac{x}{a} + \frac{y}{b} + \frac{z}{b} = 1$
Constraint: $g(a,b,c) = \frac{4}{a} + \frac{2}{b} + \frac{1}{c} = 1 \Rightarrow \nabla g = \langle -\frac{4}{a^2}, -\frac{2}{b^2}, -\frac{1}{c^2} \rangle$.
Volume: $V = f(a,b,c) = \frac{1}{6}abc \Rightarrow \nabla f = \frac{1}{6}\langle bc, ac, ab \rangle$.
Lagrange multipliers: $\frac{1}{6}\langle bc, ac, ab \rangle = -\lambda \langle \frac{4}{a^2}, \frac{2}{b^2}, \frac{1}{c^2} \rangle$ and $\frac{4}{a} + \frac{2}{b} + \frac{1}{c} = 1$.
Solve symmetrically: $-\dfrac{abc}{6\lambda} = \dfrac{4}{a} = \dfrac{2}{b} = \dfrac{1}{c}$.
Substitute in constraint: $\dfrac{3}{c} = 1 \Rightarrow \boxed{a = 12, \quad b = 6, \quad c = 3.}$

Problem 7 (Class 26, 4 pts:1,1,2)



a) Let $P = (2,5)$ and area $= A = xy$.
We have $\frac{\partial A}{\partial x} = y$ and $\frac{\partial A}{\partial y} = x \Rightarrow \left(\frac{\partial A}{\partial x}\right)_P = 5$ and $\left(\frac{\partial A}{\partial y}\right)_P = 2$.
$\boxed{\Rightarrow \Delta A \approx 5\,\Delta x + 2\,\Delta y.}$

b) $A(2,5) = 10$. $\Delta A \approx 5 \cdot .1 + 2 \cdot .2 = .9 \Rightarrow \boxed{A \approx 10.9.}$

c) The rectangle marked A_0 is the original area 10.
A_1 is the area corresponding to $2\,\Delta x$ in the approximation formula.
A_2 is the area corresponding to $5\,\Delta y$ in the approximation formula.
$A_3 = \Delta x \cdot \Delta y$ is the error in the approximation.
Since A_3 is a product of the small values the error is tiny.