18.02a Problem Set 8 Solutions

Part II (22 points)

Problem 1 (Class 30, 7 pts:1,2,2,2)

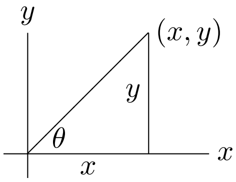


Fig. 1

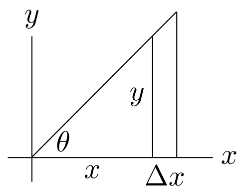


Fig. 2

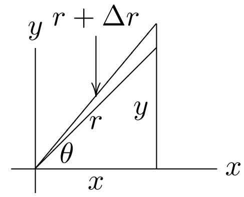


Fig. 3

a) $y = x\tan\theta$ (see Fig. 1). Area $= w = \frac{1}{2}xy = \frac{1}{2}x^2\tan\theta$.

$$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_\theta = x\tan\theta = y, \text{ and } \left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}x^2\sec^2\theta.$$

b) As before, $y = x\tan\theta$ and $w_x = \frac{1}{2}y$, $w_y = \frac{1}{2}x$.

$$\left(\frac{\partial w}{\partial x}\right)_\theta = w_x\left(\frac{\partial x}{\partial x}\right)_\theta + w_y\left(\frac{\partial y}{\partial x}\right)_\theta = \frac{1}{2}y + \frac{1}{2}x\tan\theta = x\tan\theta = y,$$

$$\left(\frac{\partial w}{\partial \theta}\right)_x = w_x\left(\frac{\partial x}{\partial \theta}\right)_x + w_y\left(\frac{\partial y}{\partial \theta}\right)_x = 0 + \frac{1}{2}x^2\sec^2\theta = \frac{1}{2}x^2\sec^2\theta.$$

c) $dw = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$, $\quad dy = \tan\theta\,dx + x\sec^2\theta\,d\theta$.

Eliminate dy from the equation for dw.

$dw = \frac{1}{2}y\,dx + \frac{1}{2}x(\tan\theta\,dx + x\sec^2\theta\,d\theta) = (\frac{1}{2}y + \frac{1}{2}x\tan\theta)dx + (\frac{1}{2}x^2\sec^2\theta)d\theta.$

$$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_\theta = \frac{1}{2}y + \frac{1}{2}x\tan\theta = y, \quad \text{and} \quad \left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}x^2\sec^2\theta.$$

d) If we fix θ and vary x then (see Fig. 2)

$\Delta w =$ area of trapezoidal strip at right $= \Delta x \cdot \frac{1}{2}(y + y + \Delta y) = y\Delta x + \frac{1}{2}\Delta x \cdot \Delta y \approx y\Delta x$.

(We ignore second order terms.) $\Rightarrow \dfrac{\Delta w}{\Delta x} \approx y \Rightarrow \left(\dfrac{\partial w}{\partial x}\right)_\theta = y.$

If we fix x and vary θ then (see Fig. 3) $\Delta w =$ area of thin wedge.

The angle of the wedge is $\Delta\theta$ and $\Delta w = \frac{1}{2}r(r+\Delta r)\sin(\Delta\theta) \approx \frac{1}{2}r(r+\Delta r)\Delta\theta \approx \frac{1}{2}r^2\Delta\theta$.

(Here, we've used $\sin x \approx x$ and then dropped second order terms.)

$$\Rightarrow \frac{\Delta w}{\Delta\theta} \approx \frac{1}{2}r^2 = \frac{1}{2}x^2\sec^2\theta \Rightarrow \left(\frac{\partial w}{\partial \theta}\right)_x = \frac{1}{2}x^2\sec^2\theta.$$

Problem 2 (Class 30, 4 pts)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}.$$

$$\begin{aligned}
\Rightarrow \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial s\,\partial r}\frac{\partial s}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial r\,\partial s}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial s^2}\frac{\partial s}{\partial x}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial s\,\partial r}\right) + \left(\frac{\partial^2 u}{\partial r\,\partial s} + \frac{\partial^2 u}{\partial s^2}\right) \\
&= \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s\,\partial r} + \frac{\partial^2 u}{\partial s^2}.
\end{aligned}$$

(continued)

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial u}{\partial r}(-c) + \frac{\partial u}{\partial s}(c).$$

$$\begin{aligned}
\Rightarrow \frac{\partial^2 u}{\partial t^2} &= -c\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) + c\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial s}\right) \\
&= -c\left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial t} + \frac{\partial^2 u}{\partial s\,\partial r}\frac{\partial s}{\partial t}\right) + c\left(\frac{\partial^2 u}{\partial r\,\partial s}\frac{\partial r}{\partial t} + \frac{\partial^2 u}{\partial s^2}\frac{\partial s}{\partial t}\right) \\
&= -c\left(\frac{\partial^2 u}{\partial r^2}(-c) + \frac{\partial^2 u}{\partial s\,\partial r}(c)\right) + c\left(\frac{\partial^2 u}{\partial r\,\partial s}(-c) + \frac{\partial^2 u}{\partial s^2}(c)\right) \\
&= c^2\frac{\partial^2 u}{\partial r^2} - 2c^2\frac{\partial^2 u}{\partial s\,\partial r} + c^2\frac{\partial^2 u}{\partial s^2}.
\end{aligned}$$

$$\Rightarrow \frac{\partial^2 u}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s\,\partial r} + \frac{\partial^2 u}{\partial s^2} - \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial s\,\partial r} - \frac{\partial^2 u}{\partial s^2} = 4\frac{\partial^2 u}{\partial s\,\partial r}. \quad \text{QED}$$

Problem 3 (Class 30, 3 pts)
We do this in small steps.

Original formula: $\left(\frac{\partial U}{\partial p}\right)_T + T\left(\frac{\partial V}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial p}\right)_T = 0.$ $\quad(\star\star)$

Step 1. Chain rule:

$$\left(\tfrac{\partial w}{\partial u}\right)_v = \left(\tfrac{\partial w}{\partial x}\right)_y\left(\tfrac{\partial x}{\partial u}\right)_v + \left(\tfrac{\partial w}{\partial y}\right)_x\left(\tfrac{\partial y}{\partial u}\right)_v, \qquad \left(\tfrac{\partial w}{\partial v}\right)_u = \left(\tfrac{\partial w}{\partial x}\right)_y\left(\tfrac{\partial x}{\partial v}\right)_u + \left(\tfrac{\partial w}{\partial y}\right)_x\left(\tfrac{\partial y}{\partial v}\right)_u.$$

Step 2. Write in matrix form:

$$\left(\left(\tfrac{\partial w}{\partial u}\right)_v,\ \left(\tfrac{\partial w}{\partial v}\right)_u\right) = \left(\left(\tfrac{\partial w}{\partial x}\right)_y,\ \left(\tfrac{\partial w}{\partial y}\right)_x\right)\begin{pmatrix} \left(\tfrac{\partial x}{\partial u}\right)_v & \left(\tfrac{\partial x}{\partial v}\right)_u \\ \left(\tfrac{\partial y}{\partial u}\right)_v & \left(\tfrac{\partial y}{\partial v}\right)_u \end{pmatrix}.$$

Step 3. Decide which variables are (x, y) and which are (u, v):
Want U, V to become the independent variables, i.e. want U, V in denominator of Jacobian matrix: $\Rightarrow\ (x, y) \leftrightarrow (p, T)$, and $(u, v) \leftrightarrow (U, V)$.

Step 4. Substitute into formula in step 2:

$$\left(\left(\tfrac{\partial w}{\partial U}\right)_V,\ \left(\tfrac{\partial w}{\partial V}\right)_U\right) = \left(\left(\tfrac{\partial w}{\partial p}\right)_T,\ \left(\tfrac{\partial w}{\partial T}\right)_p\right)\begin{pmatrix} \left(\tfrac{\partial p}{\partial U}\right)_V & \left(\tfrac{\partial p}{\partial V}\right)_U \\ \left(\tfrac{\partial T}{\partial U}\right)_V & \left(\tfrac{\partial T}{\partial V}\right)_U \end{pmatrix}.$$

In this case there is no confusion, so for ease of notation we'll let $w_U = \left(\frac{\partial w}{\partial U}\right)_V$ etc.

Step 5. Call the matrix A, find A^{-1}: $\quad A^{-1} = \frac{1}{|A|}\begin{pmatrix} T_V & -p_V \\ -T_U & p_U \end{pmatrix}.$

Step 6. Choose various w to get all the pieces in formula $(\star\star)$:

$$w = U \Rightarrow \left(\left(\tfrac{\partial U}{\partial p}\right)_T,\ \left(\tfrac{\partial U}{\partial T}\right)_p\right) = \left(\left(\tfrac{\partial U}{\partial U}\right)_V,\ \left(\tfrac{\partial U}{\partial V}\right)_U\right)\cdot A^{-1} = (1,\ 0)\cdot A^{-1} = \tfrac{1}{|A|}\left(T_V,\ -p_V\right).$$

$$w = V \Rightarrow \left(\left(\tfrac{\partial V}{\partial p}\right)_T,\ \left(\tfrac{\partial V}{\partial T}\right)_p\right) = \left(\left(\tfrac{\partial V}{\partial U}\right)_V,\ \left(\tfrac{\partial V}{\partial V}\right)_U\right)\cdot A^{-1} = (0, 1)\cdot A^{-1} = \tfrac{1}{|A|}\left(-T_U,\ p_U\right).$$

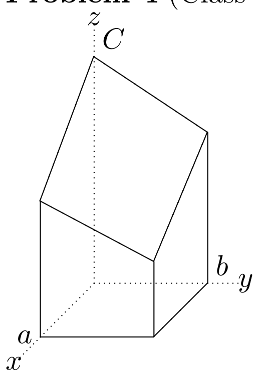
I.e. $\left(\frac{\partial U}{\partial p}\right)_T = \frac{1}{|A|}\left(\frac{\partial T}{\partial V}\right)_U, \quad \left(\frac{\partial V}{\partial p}\right)_T = -\frac{1}{|A|}\left(\frac{\partial T}{\partial U}\right)_V, \quad \left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{|A|}\left(\frac{\partial p}{\partial U}\right)_V.$

Step 7. Substitute into the law $(\star\star)$:

$$\frac{1}{|A|}\left(\left(\tfrac{\partial T}{\partial V}\right)_U + T\left(\tfrac{\partial p}{\partial U}\right)_V - p\left(\tfrac{\partial T}{\partial U}\right)_V\right) = 0 \ \Rightarrow\ \boxed{\left(\tfrac{\partial T}{\partial V}\right)_U + T\left(\tfrac{\partial p}{\partial U}\right)_V - p\left(\tfrac{\partial T}{\partial U}\right)_V = 0.}$$

(continued)

Problem 4 (Class 31, 3 pts)



Equation of top plane: $z = Ax + By + C$.

Vol $= \iint_R z\,dA = \int_0^a \int_0^b Ax + By + C\,dy\,dx$.

Inner: $Axy + By^2/2 + Cy|_0^b \quad = \quad Abx + Bb^2/2 + Cb$.

Outer: $\int_0^a Abx + Bb^2/2 + Cb\,dx \quad = \quad Aa^2b/2 + Bb^2a/2 + Cab$
$\quad = \quad ab(aA/2 + bB/2 + C)$.

Length of four legs: C, $Aa + C$, $Bb + C$, $Aa + Bb + C$

$\Rightarrow$ average $= C + Aa/2 + Bb/2$.

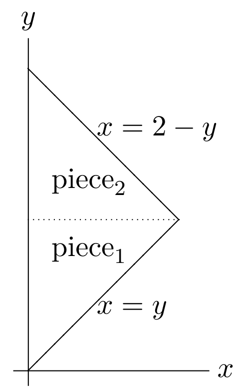
$\Rightarrow$ volume = (area base)$\cdot$ (average of legs).

Problem 5 (Class 31, 3 pts)
The original integral is over the triangular region shown.

Changing the order of integration we need to break it into two pieces.

Piece 1: $y : 0$ to 1; For fixed y, $x : 0$ to y.

Piece 2: $y : 1$ to 2; For fixed y, $x : 0$ to $2 - y$.



$$\Rightarrow \text{ integral} = \int_0^1 \int_0^y \frac{x}{y}\,dx\,dy + \int_1^2 \int_0^{2-y} \frac{x}{y}\,dx\,dy.$$

$$\text{Inner}_1 = \left.\frac{x^2}{2y}\right|_0^y = \frac{y}{2}.$$

$$\text{Inner}_2 = \left.\frac{x^2}{2y}\right|_0^{2-y} = \frac{(2-y)^2}{2y} = \frac{2}{y} - 2 + \frac{y}{2}.$$

$$\text{Outer}_1 = \left.\frac{y^2}{4}\right|_0^1 = \frac{1}{4}.$$

$$\text{Outer}_2 = 2\ln y - 2y + \left.\frac{y^2}{4}\right|_1^2 = 2\ln 2 - 4 + 1 + 2 - \frac{1}{4}.$$

Integral = outer$_1$ + outer$_2$ = $\boxed{2\ln 2 - 1.}$

You could do this integral in the original order, but it involves more difficult integrals.

Problem 6 (Class 31, 2 pts)
Since the inner integral has dx it is over x –it should not have x as a limit of integration.