18.01A Topic 1: Linear and quadratic approximations.
Read: SN: A.

Vocabulary: Linear approximation = linearization
Quadratic approximation
Geometric series
Binomial theorem

Basic idea: If h is small then h^2 is really small and h^3 is really, really small.

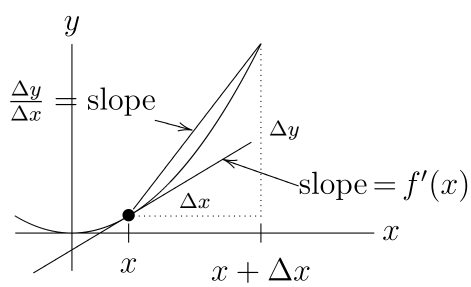
Example: Suppose $f(x) = 3 + 4x + 5x^2 + 7x^3$.
Then for x small, $f(x) \approx 3 + 4x$ (linear approximation).
I.e. we can ignore the higher powers of x.

More accurate approximation: $f(x) \approx 3 + 4x + 5x^2$ (quadratic approx.)
Notice that
1. $f(0) = 0$.
2. $f'(x) = 4 + 2 \cdot 5x + 3 \cdot 7x^2 \Rightarrow f'(0) = 4$.
4. $f''(x) = 2 \cdot 5 + 3 \cdot 2 \cdot 7x \Rightarrow f''(0) = 2 \cdot 5$.

$\Rightarrow$ linear approximation is: $f(x) \approx f(0) + f'(0)x$ for $x \approx 0$.
and quadratic approximation is: $f(x) \approx f(0) + f'(0)x + \dfrac{1}{2}f''(0)x^2$ for $x \approx 0$.

To see why these are the *best* approximations we turn to calculus.
While we're at it we'll work near an arbitrary base point $x = a$.

Basic idea:
For $y = f(x)$, $f'(x) \approx \frac{\Delta y}{\Delta x} \Rightarrow \Delta y \approx f'(x)\,\Delta x$.
$\Leftrightarrow f(x) - f(a) \approx f'(x)\,(x - a)$.
(I.e the tangent line approximates the graph.)



Basic linear formulas:
(A2) $f(x) \approx f(a) + f'(a)(x - a)$ for $x \approx a$.
(A4) $1/(1 - x) \approx 1 + x$ for $x \approx 0$.
(A5) $(1 + x)^r \approx 1 + rx$ for $x \approx 0$.
(A6) $\sin x \approx x$ for $x \approx 0$.

We can prove A4-6 using A2.

Examples:
1. Approximate $f(x) = (1 + x)^{99}(1 + 3x)^{77}$ for $x \approx 0$.
answer: $f(x) \approx (1 + 99x)(1 + 77 \cdot 3x) = 1 + 330x + 99 \cdot 231x^2 \approx 1 + 330x$.

2. Aproximate $f(x) = 1/(1 - \sin x)^2$ for $x \approx 0$.
answer: Since $\sin x \approx 0$ when $x \approx 0$ (A4) gives
$f(x) \approx (1 + \sin x)^2 \approx (1 + x)^2 = 1 + 2x + x^2 \approx 1 + 2x$.

(continued)

Basic quadratic formulas:
(A13) $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$ for $x \approx a$.
(A9) $\frac{1}{1-x} \approx 1 + x + x^2$ for $x \approx 0$.
(A10) $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$ for $x \approx 0$.
(A11) $\sin x \approx x$ for $x \approx 0$.
(A12) $\cos x \approx 1 - \frac{x^2}{2}$ for $x \approx 0$.

We can prove A9-12 using A13.

Examples: (quadratic)
Q1. $\sqrt{a+bx} = \sqrt{a}\sqrt{1 + \frac{b}{a}x} \approx \sqrt{a}(1 + \frac{1}{2}\frac{b}{a}x - \frac{1}{8}(\frac{b}{a}x)^2)$ (for $x \approx 0$).

Q2. $\tan\theta = \frac{\sin\theta}{\cos\theta}$, (near 0) $\approx \frac{\theta}{1-\theta^2/2} \approx \theta(1+\theta^2/2) \approx \theta$.

Examples:
2A-1. Find the linearization of $\sqrt{a+bx}$ in two ways. First by using formula (A2) and second using the basic formulas and algebra.
__**answer:**__ i) Give the function a name: $f(x) = \sqrt{a+bx}$.
Find the pieces of (A2):
$f(0) = \sqrt{a}$; $f(x) = \sqrt{a+bx} \Rightarrow f'(x) = b/\sqrt{a+bx} \Rightarrow f'(0) = b/\sqrt{a}$.
Use (A2): $f(x) \approx \sqrt{a} + \frac{b}{\sqrt{a}}x$, for $x \approx 0$.

ii) This is done in example Q1 above, simply ignore the quadratic term.

2A-2. Same as exercise 1 for $f(x) = \frac{1}{a+bx}$.
__**answer:**__ i) Find the pieces of (A2):
$f(0) = \frac{1}{a}$; $f'(x) = -\frac{b}{(a+bx)^2} \Rightarrow f'(0) = -\frac{b}{a^2}$.
Use (A2): $f(x) \approx \frac{1}{a} - \frac{b}{a^2}x$, for $x \approx 0$.

2A-8. Find the quadratic approximation for $f(x) = \frac{1}{1-x}$ for $x \approx 1/2$.

__**answer:**__ Find the pieces for (A13) (here, $a = \frac{1}{2}$):

$f(\frac{1}{2}) = 2$; $f'(x) = \frac{1}{(1-x)^2} \Rightarrow f'(\frac{1}{2}) = 4$; $f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(\frac{1}{2}) = 16$.
Use (A13): $f(x) \approx 2 + 4(x - \frac{1}{2}) + 8(x - \frac{1}{2})^2$.

(continued)

Same problem, finding the answer using algebra:
Let $y = f(x)$.
Let $u = x - \frac{1}{2}$, (so $x \approx \frac{1}{2} \Leftrightarrow u \approx 0$).
$\Rightarrow\; y = \dfrac{1}{1/2 - u} = \dfrac{2}{1-2u} \approx 2(1 + 2u + 4u^2) = 2 + 4(x - \dfrac{1}{2}) + 8(x - \dfrac{1}{2})^2.$
(The first approximation comes using (A9).)

Example from special relativity (example 3 in notes §A)
$m = m_0 c/\sqrt{c^2 - v^2}$, what v needed to produce 1% increase in mass?
Want $m/m_0 = 1.01 = c/\sqrt{c^2 - v^2} = (1 - (v/c)^2)^{-1/2} \approx 1 + \frac{1}{2}(v/c)^2$.
Let $u = v/c$, $1.01 = 1 + \frac{1}{2}u^2 \;\Rightarrow\; .02 = u^2 \;\Rightarrow\; u \approx \frac{1}{7} \;\Rightarrow\; v \approx 27000$ mi/sec.

Algebraic substitution rules:
1. Can substitute a linear (quadratic) approx for any factor or divisor as long as they have a constant term.
2. Once you make a linear substitution you can never recover the quadratic approximation.

Examples: (why we need to have a constant term)
1. $\frac{x(1+x)}{x(2+x)} \not\approx \frac{x}{2x}$.
2. $\dfrac{\ln(1+x)}{x e^x} \not\approx \dfrac{x}{x} = 1.$
Instead $= \frac{\ln(1+x)/x}{e^x} \approx \frac{1-x/2}{1+x} \approx (1 - x/2)(1-x) \approx 1 - 3x/2$.
(Note: this would be hard to do by differentiation.)

Example: (why we can't get the quad. approx after a linear substitution)
$f(x) = (1 + x + x^2 + x^3)(1 + 2x + 3x^2)$
Quad. approx near 0: $f(x) \approx (1 + x + x^2)(1 + 2x + 3x^2) \approx 1 + 3x + 6x^2$
If first made linear approx: $f(x) \approx (1+x)(1+2x) = 1 + 3x + 2x^2$ which is not *THE* quadratic approx. of $f(x)$

The exponential function $e^t \approx 1 + t + t^2/2$ for t near 0.
Proof: Let $f(t) = e^t$ then $f(t) = f'(t) = f''(t) = e^t$.
$\Rightarrow\; f(0) = f'(0) = f''(0) = 1.$
$\Rightarrow\; f(t) \approx 1 + t + t^2/2$ for $t \approx 0$.

Examples:
Suppose you have \$1000 in bank at 2% continuous interest. Approximately how much money is in the bank after 1 year? After 2 years?
answer: Balance $= f(t) = 1000e^{.02t} \approx 1000(1 + .02t + (.02t)^2/2)$.
$f(1) \approx 1000(1 + .02 + .0002) = 1020.20.$
$f(2) \approx 1000(1 + .04 + .0008) = 1040.80.$