18.01A Topic 10: Integration by parts, numerical integration.
Read: TB: 10.7, 10.9.

Integration by parts:
Main idea is product rule:

$$(uv)' = uv' + vu' \leftrightarrow d(uv) = u\,dv + v\,du.$$

$\Rightarrow\ u\,dv = d(uv) - v\,du$

Integrating $\Rightarrow$ $\boxed{\int u\,dv = uv - \int v\,du}$

Examples: (I suggest you learn to use the following format.)

1. $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$ $\quad \boxed{\begin{array}{ll} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{array}}$

2. $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C.$ $\quad \boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}\,dx & v = x \end{array}}$

3. $\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx.$ $\quad \boxed{\begin{array}{ll} u = x^2 & dv = e^x\,dx \\ du = 2x\,dx & v = e^x \end{array}}$

Use parts again (we do side work on the term in question):
$\int 2xe^x\,dx = 2xe^x - 2e^x$. (already did this in previous example).
Answer: $x^2e^x - 2xe^x + 2e^x + C$.

4. $\int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx$ $\quad \boxed{\begin{array}{ll} u = \cos x & dv = e^x\,dx \\ du = -\sin x\,dx & v = e^x \end{array}}$

$= e^x\cos x + e^x\sin x - \int e^x\cos x\,dx.$ $\quad \boxed{\begin{array}{ll} u = \sin x & dv = e^x\,dx \\ du = \cos x\,dx & v = e^x \end{array}}$

$\Rightarrow\ 2\int e^x\cos x\,dx = e^x\cos x + e^x\sin x.$
Answer: $\frac{1}{2}e^x(\cos x + \sin x)$.

(continued)

Inverse trig functions: We know $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$.

(Do this again in a minute.)

Using it we compute:

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = \sin^{-1}x$	$dv = dx$
$du = \frac{1}{\sqrt{1-x^2}}\,dx$	$v = x$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C.$$

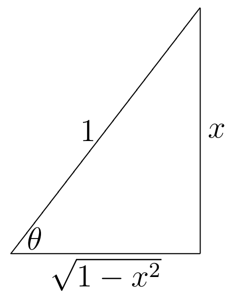
Computing the derivative of $\sin^{-1}x$:

Note: $\sin(\sin^{-1}x) = x$

Take deriv of both sides (use chain rule):

$\cos(\sin^{-1}x)\frac{d}{dx}\sin^{-1}x = 1.$

$\Rightarrow \frac{d}{dx}\sin^{-1}x = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}.$



A recursive formula:

Claim: $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$

Proof: By parts:

$u = \sin^{n-1}x$	$dv = \sin x\,dx$
$du = (n-1)\sin^{n-2}x\cos x\,dx$	$v = -\cos x$

$$\Rightarrow \int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x - \sin^n x\,dx.$$

$\Rightarrow n\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx.$ ■

This is recursive because:

$$\int \sin^5 x\,dx = -\tfrac{1}{5}\sin^4 x\cos x + \tfrac{4}{5}\int \sin^3 x\,dx$$
$$= -\tfrac{1}{5}\sin^4 x\cos x - \tfrac{4}{5}\cdot\tfrac{1}{3}\sin^2 x\cos x + \tfrac{4}{5}\tfrac{2}{3}\int \sin x\,dx$$
$$= -\tfrac{1}{5}\sin^4 x\cos x - \tfrac{4}{5}\cdot\tfrac{1}{3}\sin^2 x\cos x - \tfrac{4}{5}\tfrac{2}{3}\cos x + C.$$

(continued)

Numerical Integration:
Why?
Can use computers.
When no closed formula, e.g. $\int e^{-x^2}$, $\int \sqrt{\sin x}$, $\int \sqrt{1+\sin^2 x}$.

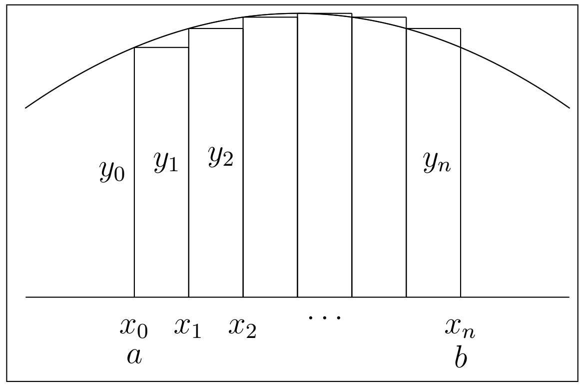
Idea: Riemann Sums.

Rectangles:
Right Riemann sum

$$\int_a^b f(x)\,dx = \text{ area } \approx \sum_{j=1}^{n} y_j\,\Delta x.$$

Left Riemann sum

$$\int_a^b f(x)\,dx = \text{ area } \approx \sum_{j=0}^{n-1} y_j\,\Delta x.$$

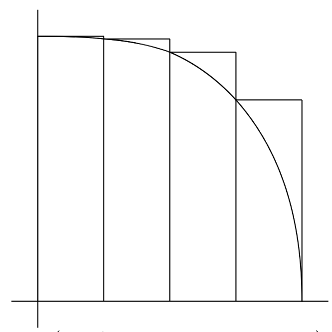


(Left Riemann Sum)

Other possibilities:
mid Riemann sum
max Riemann sum
min Riemann sum
random Riemann sum.

Example: Estimate $\int_0^1 \sqrt{1-x^3}\,dx$.
To avoid too much calculation choose $n = 4$.

j	0	1	2	3	4
x_j	0	.25	.5	.75	1
y_j	1	.992	.935	.760	0


(Left Riemann sum)

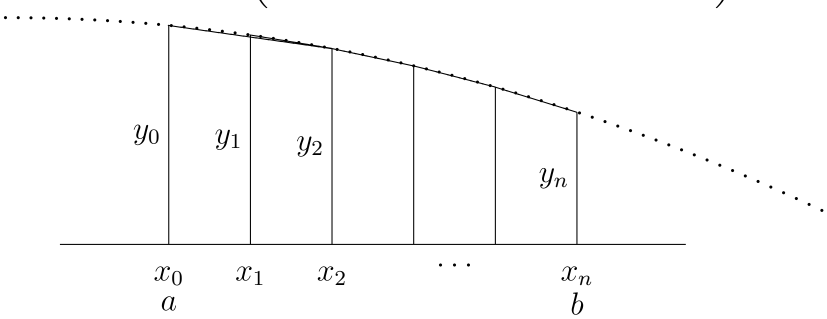
Right R.S. $= (y_1 + y_2 + y_3 + y_4)\,\Delta x \approx .67$.
Left R.S. $= (y_0 + y_1 + y_2 + y_3)\,\Delta x \approx .922$.
With $n = 100$:
Right R.S. = .836, Left R.S. = .846.
With $n = 500$:
Right R.S. = .840, Left R.S. = .842.

"True" value = .8413 (from Simpson's rule with $n = 1.6 \times 10^6$).

(continued)

Trapezoidal Rule:
Idea: Use trapezoids instead of rectangles.
Equals average of left and right Riemann sums.

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \left(\frac{y_0+y_1}{2}+\frac{y_1+y_2}{2}+\cdots+\frac{y_{n-1}+y_n}{2}\right)\Delta x\\ &= \left(\frac{1}{2}y_0+y_1+y_2+\cdots+y_{n-1}+\frac{1}{2}y_n\right)\Delta x.\end{aligned}$$



Example: (same as above) $\int_0^1 \sqrt{1-x^3}\,dx$ with $n=4$.
Integral $\approx (\frac{1}{2}\cdot 1 + .992 + .935 + .760 + \frac{1}{2}\cdot 0)\cdot .25 = .797$.

With $n=30$, integral $\approx .839$ (much faster convergence than rectangles).

Simpson's Rule:
Idea: Cap tops with parabolas instead of lines.
Derivation in book.
Must have n even: $\int_a^b f(x)\,dx \approx \frac{1}{3}(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+4y_{n-1}+y_n)\,\Delta x$.

Example: (same as above with $n=4$)
$\int_0^1 \sqrt{1-x^3}\,dx \approx \frac{1}{3}(1+4(.992)+2(.935)+4(.760)+0)\cdot .25 = .823$.

With $n=30$: $\int_0^1 \sqrt{1-x^3} \approx .840$ (fastest convergence).