18.01A Topic 11: Improper integrals.
Read: TB: 12.4, SN: INT

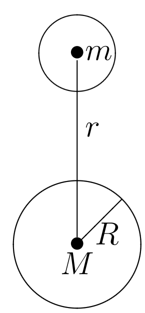
Definition: $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$
If the limit exits we say the *improper* integral converges otherwise we say it diverges.

Example: Compute $\displaystyle\int_1^\infty \frac{1}{x^n}\,dx.$ (To avoid an annoying case, assume $n \neq 1$.)

$$\text{Improper int.} = \lim_{b\to\infty}\left.\frac{x^{-n+1}}{-n+1}\right|_1^b = \lim_{b\to\infty}\frac{b^{-n+1}}{-n+1} + \frac{1}{n-1} = \begin{cases}\frac{1}{n-1} & \text{if } n>1\\ \infty & \text{if } n<1\end{cases}$$

So the integral converges if $n > 1$ and diverges to ∞ if $n < 1$.
If $n = 1$ it also diverges to ∞.

Example: (Probably don't have time to do this in class. We did some of this before, when we worked on work.)
Compute the work (energy) needed to get a mass m to a distance x from the center of the earth. (Assume $x > R$.)

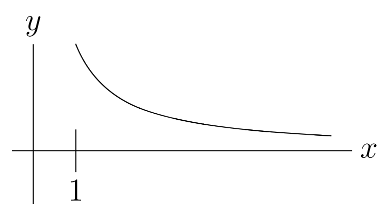


r = distance from center of earth.
R = radius of earth.
$F = \frac{GmM}{r^2} = \frac{C}{r^2}$ = force on m at distance r.
Using 'slicing and summing' we found that the work (energy) needed to get a mass m from the surface to a distance x from the center of the earth (assume $x > R$) is

$$W = \int_R^x \frac{C}{r^2}\,dr = \left.\frac{-C}{r}\right|_R^r = \frac{C}{R} - \frac{C}{x}.$$

To escape earth's gravity we let $x \to \infty \Rightarrow W \to \frac{C}{R}$, i.e. need at least $\frac{C}{R}$ units of energy to escape.

Example: Consider $f(x) = \frac{1}{x}$ for $1 \le x < \infty$.
Area under curve $= \int_1^\infty \frac{1}{x}\,dx = \ln x|_1^\infty = \infty$.
Volume of revolution around x-axis:
$\int_1^\infty \pi(1/x)^2\,dx = \pi$ is finite.



Comparison test: Suppose $0 \le f(x) \le g(x)$
$\int_a^\infty g(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ converges.
$\int_a^\infty f(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ diverges.

Examples:
$\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx < \int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges.
$\int_0^\infty \mathrm{e}^{-x^2}\,dx = \int_0^1 \mathrm{e}^{-x^2}\,dx + \int_1^\infty \mathrm{e}^{-x^2}\,dx < \int_0^1 \mathrm{e}^{-x^2}\,dx + \int_1^\infty \mathrm{e}^{-x}\,dx$ converges.

(continued)

This is a pain. We can do better.

Limit comparison: Assume f, g are positive functions.
If $\lim_{x\to\infty} \dfrac{f}{g} = c$ and $c \neq 0, \infty$ then both improper integrals converge or both diverge.

Notes: 1. If $c \neq 0, \infty$ we write $f \sim g$.
2. Also called **Asymptotic comparison**.

Example: If $f = \frac{1}{\sqrt{x^3-1}}$, $g = \frac{1}{x^{3/2}}$ then $\lim_{x\to\infty} \dfrac{f}{g} = 1$.
Thus, since $\int_2^\infty \frac{1}{x^{3/2}}\,dx$ converges so does $\int_2^\infty \frac{1}{\sqrt{x^3-1}}\,dx$.

Note: We need the lower limit in the integral > 1 so $g(x)$ is defined on the entire interval we integrate over.

"Proof" of limit comparison:
$\lim \frac{f}{g} = c \;\Rightarrow\;$ there is an a such that $\frac{c}{2}g(x) < f(x) < 2cg(x)$ for $x > a$.
Thus $\frac{c}{2}\int_a^\infty g(x)\,dx < \int_a^\infty f(x)\,dx < 2c\int_a^\infty g(x)\,dx$. This shows both integrals converge or both diverge. QED

Other improper integrals:

Example: $\int_0^1 \frac{1}{x^{1/3}}\,dx$ This is improper because $\frac{1}{x^{1/3}} = \infty$ when $x = 0$.

We define $\displaystyle\int_{0^+}^1 \frac{1}{x^{1/3}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^{1/3}}\,dx.$

$$\Rightarrow \text{ Integral} = \lim_{a\to 0^+} \frac{3}{2}x^{2/3}\Big|_a^1 = \lim_{a\to 0^+} \frac{3}{2} - \frac{3}{2}a^{2/3} = \frac{3}{2}.$$

$\Rightarrow$ the integral converges.

P test:

$$\int_1^\infty x^p\,dx \to \begin{cases} p < -1 & \text{converges} \\ p \geq -1 & \text{diverges} \end{cases} \qquad \int_{0^+}^1 x^p\,dx \to \begin{cases} p > -1 & \text{converges} \\ p \leq -1 & \text{diverges} \end{cases}$$

Always think about these. It is sometimes written as:

$$\int_1^\infty \frac{1}{x^p}\,dx \to \begin{cases} p > 1 & \text{converges} \\ p \leq 1 & \text{diverges} \end{cases} \qquad \int_{0^+}^1 \frac{1}{x^p}\,dx \to \begin{cases} p < 1 & \text{converges} \\ p \geq 1 & \text{diverges} \end{cases}$$

Comparison works for these types of improper integrals also.

Example: $\int_{0^+}^1 \frac{1}{\sqrt{x(x+1)}}\,dx$ converges since $\frac{1}{\sqrt{x(x+1)}} \leq \frac{1}{\sqrt{x}}$

Be careful –at first glance you might think the appropriate p for comparison is $p = 1$.

Example: (change of variable)

Does $\displaystyle\int_0^{1^-} \frac{1}{\sqrt{1-x^3}}\,dx$ converge? (Improper at 1.)

Substitute $u = 1 - x \;\Rightarrow\;$ Integral $= \displaystyle\int_{0^+}^1 \frac{1}{\sqrt{u}\sqrt{u^2-3u+3}}\,du$

Converges by comparison with $1/\sqrt{u}$.