18.01A Topic 12: Infinite series, harmonic series convergence tests.
Read: TB: 13.1 and 13.2 quickly, 13.3 to top p.442, 13.5 to p.453, 13.6 to p.457

MUST get comfortable with $\sum$ notation.

Definition of series (= sum):
$a_0 + a_1 + a_2 + \ldots + a_n$ (finite series).
$a_0 + a_1 + a_2 + \ldots + a_n + \ldots$ (infinite series).

Sigma notation: $\displaystyle\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots.$

Note: The index n could be another letter, e.g. i, j. It's like the x in an integral.

Partial sums: $S_N = a_0 + a_1 + \ldots + a_N$ is called the N^{th} partial sum.

Definition of convergent series:
If $\lim_{N\to\infty} S_N = S$ exists the series converges to the sum S, otherwise it diverges.

Note If the limit is ∞ we say the series diverges to ∞.

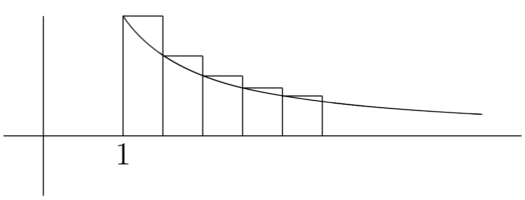
Example: Geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$. For $|r| < 1$ this converges to $\dfrac{1}{1-r}$. For $|r| \geq 1$ the geometric series diverges. (Proof: $S_N = \dfrac{1-r^{N+1}}{1-r}$.)

Example: Harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$

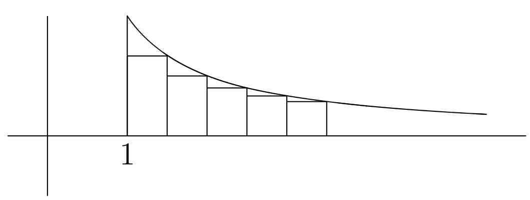
Claim: This diverges to ∞.

Proof 1: (This is also in the book so won't do in class.)
sum $= (1) + (\frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \ldots$
Each grouping of terms is $> \frac{1}{2}$, for instance the one starting with $\frac{1}{5}$ has 4 terms all $\geq \frac{1}{8}$. Continuing by taking twice as many terms in each successive group produces an infinite sum of groups $> \frac{1}{2} \Rightarrow$ diverges to ∞.

Proof 2 (integral test):
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx = \ln x|_1^{\infty} = \infty.$ (The inequality follows because the sum is a left Riemann sum that overestimates the area under the curve.)



Left Riem. sum overest. integral



Right Riem. sum underest. integral

(continued)

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ converges.

Proof: Series = right Riem. sum $< \int_0^\infty \frac{1}{x^2}\,dx = 1$.

> **Integral Test**:
> If $f(x)$ is decreasing and $\lim_{x\to\infty} f(x) = 0$ then
> $\displaystyle\sum_{n=n_0}^{\infty} f(n)$ and $\displaystyle\int_a^\infty f(x)\,dx$ either both converge or both diverge.

Proof: Left Riemann sum $= \sum_{n=n_0}^{\infty} f(n) > \int_{n_0}^\infty f(x)\,dx >$ right Riemann sum $= \sum_{n=n_0+1}^{\infty} f(n)$.

N.B. the hypotheses that $f(x)$ is decreasing and goes to 0.

Example: (**p-test**) Does $\displaystyle\sum_{n=7}^{\infty} \frac{1}{n^p}$ converge or diverge?

The function $f(x) = \frac{1}{x^p}$ satisfies the hypotheses of the integral test.

Since $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ so does the sum.

Likewise it diverges for $p \le 1$.

Example: Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ converge?

answer: $\int_2^\infty \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\,d(\ln x) = \ln\ln x|_2^\infty = \infty \Rightarrow$ diverges.

(Of course, must check that $\frac{1}{x\ln x}$ is decreasing.)

> **Comparison test**:
> Assume $0 \le f(n) \le g(n)$
>
> If $\displaystyle\sum_{n=n_0}^{\infty} g(n)$ converges then so does $\displaystyle\sum_{n=n_0}^{\infty} f(n)$.
>
> If $\displaystyle\sum_{n=n_0}^{\infty} f(n)$ diverges then so does $\displaystyle\sum_{n=n_0}^{\infty} g(n)$.

Example: $\sum \frac{1}{n^2+1} < \sum \frac{1}{n^2}$ converges.

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Asymptotic comparison test:
Assume a_n, b_n are positive.
If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c \neq 0, \infty$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Proof: For large n (say for $n \geq N$) $\frac{a_n}{b_n} \approx c \;\Rightarrow\;$ for large n: $\frac{1}{2}c\,b_n < a_n < 2\,c\,b_n$.

So $\displaystyle\sum_{n=N}^{\infty} a_n$ converges $\Rightarrow \displaystyle\sum \frac{1}{2}c\,b_n$ also converges $\Rightarrow \displaystyle\sum b_n$ converges.

Likewise $\displaystyle\sum_{n=N}^{\infty} b_n$ converges $\Rightarrow \displaystyle\sum 2c\,b_n$ converges $\Rightarrow \displaystyle\sum a_n$ converges.

Note: $\displaystyle\sum_{n=N}^{\infty} a_n$ converges $\Leftrightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Examples: Do the following converge or diverge?

1. $\displaystyle\sum \frac{2}{n^2+n}$ (Converges –compare with $\sum \frac{1}{n^2}$.)
2. $\displaystyle\sum \frac{n^2+3}{1000n^3}$ (Diverges –asymptotically compare with $\sum \frac{1}{n}$.)
3. $\displaystyle\sum \frac{1}{(n+3)^2}$ (Converges –compare with $\sum \frac{1}{n^2}$.)
4. $\displaystyle\sum \frac{n}{\sqrt{n^2+2}}$ (Diverges –asymptotically compare with $\sum 1$.)
5. $\displaystyle\sum \frac{\tan^{-1} n}{n^3}$ (Converges –asymptotically compare with $\sum \frac{1}{n^3}$ –recall $\tan^{-1} x$ is bounded between $-\pi/2$ and $\pi/2$.)

Theorem: $\sum a_n$ converges $\Rightarrow a_n \to 0$

Proof:

$$\begin{array}{ccccc} S_{n+1} & = & S_n & + & a_n \\ \downarrow & & \downarrow & & \downarrow \\ S & & S & & 0 \end{array}$$

Note: Converse of this is false, e.g. the harmonic series.

Example: (Telescoping series) $\displaystyle\sum \frac{1}{n(n+1)}$ converges by comparison to $\sum \frac{1}{n^2}$.

In this case we can actually compute the sum: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \;\Rightarrow\;$ telescoping series
$S_N = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{N} - \frac{1}{N+1}) = 1 - \frac{1}{N+1} \to 1$