18.01A Topic 13: Geometric series, power series, ratio test.
Read: TB: 13.7 to middle p.463, 13.8, 14.2

Geometric Series:

Ratio r, geometric series $= 1 + r + r^2 + r^3 + \ldots = \sum_{n=0}^{\infty} r^n$.

Punch line: Converges to $\frac{1}{r-1}$ if $|r| < 1$.

Diverges if $|r| > 1$.

Examples:

1) $\sum_{n=0}^{\infty} (\frac{1}{2})^n = \frac{1}{1-1/2} = 2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

2) $\sum_{n=0}^{\infty} (-\frac{1}{3})^n = \frac{1}{1-(-1/3)} = \frac{3}{4} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

3) $\sum \frac{1}{(1+(n-1)/n)^n}$ (Converges –asymptotically compare with $\sum 2^{-n}$, or directly compare with $\sum 1.5^n$.)

Variations

$$\sum_{n=5}^{\infty} (\frac{3}{4})^n == (\frac{3}{4})^5 + (\frac{3}{4})^6 + (\frac{3}{4})^7 + \ldots = (\frac{3}{4})^5 (1 + \frac{3}{4} + (\frac{3}{4})^2 + \ldots) = (\frac{3}{4})^5 (\frac{1}{1-3/4}) = (\frac{3}{4})^5 \cdot 4.$$

$\sum_{n=2}^{\infty} \pi r^n = \pi r^2 (\frac{1}{1-r})$ (provided $|r| < 1$).

$\sum_{n=0}^{\infty} (\frac{1}{2})^{2n} = \sum_{n=0}^{\infty} (\frac{1}{4})^n = \frac{4}{3}.$

Proof (of sum of geometric series)

N$^{\text{th}}$ partial sum:
$$S_N = 1 + r + r^2 + \ldots + r^N$$
$$rS_N = r + r^2 + r^3 + \ldots + r^{N+1}$$

Subtract above two equations: $\Rightarrow (1-r)S_N = 1 - r^{N+1} \Rightarrow S_N = \frac{1-r^{N+1}}{1-r}$

Taking limits: $\lim_{N\to\infty} S_N = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } r \geq 1 \\ \text{doesn't exist} & \text{if } r \leq -1 \end{cases}$

(continued)

Ratio Test
Assume: $a_n > 0$ for all n, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

$L < 1 \;\Rightarrow\; \sum a_n$ converges.
$L > 1 \;\Rightarrow\; \sum a_n$ diverges.
$L = 1$ no conclusion.
No limit $\Rightarrow$ no conclusion.

Examples:

1) $\displaystyle\sum_1^\infty \frac{2^n}{n^3} \;\Rightarrow\; L = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)^3}{2^n/n^3} = \lim 2\frac{n^3}{(n+1)^3} = 2 \;\Rightarrow\;$ diverges.

2) $\displaystyle\sum_1^\infty \frac{2^n}{n!} \;\Rightarrow\; L = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim 2/(n+1) = 0 \;\Rightarrow\;$ converges.

3) $\displaystyle\sum_1^\infty \frac{1}{n^2} \;\Rightarrow\; L \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \;\Rightarrow\;$ test fails.

(For this one we already know that it converges by the integral test.)

Power Series: Given $\displaystyle\sum_{n=0}^\infty a_n x^n$ for what values of x does it converge?

Theorem: If $\displaystyle\sum_{n=0}^\infty |a_n x^n|$ converges then so does $\displaystyle\sum_{n=0}^\infty a_n x^n$.

If the series with absolute value signs converges we say it's **absolutely convergent**.
If it converges but not absolutely we say it's **conditionally convergent**.

Ratio Test examples for power series:

1) $\displaystyle\sum_1^\infty \frac{x^n}{n}$ ratio $= \dfrac{|x^{n+1}/(n+1)|}{|x^n/n|} = |x| \cdot \dfrac{n}{n+1}$.

Taking the limit: $\lim_{n\to\infty} |x| \cdot \dfrac{n}{n+1} = |x|$

$\Rightarrow$ series converges for $|x| < 1$, diverges for $|x| > 1$.
(For $|x| = 1$ we need to look more carefully.)

2) $\displaystyle\sum_1^\infty \frac{x^n}{n!}$ ratio $= \dfrac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = |x| \cdot \dfrac{1}{n+1}$.

Taking the limit: $\lim_{n\to\infty} |x| \cdot \dfrac{1}{n+1} = 0 \quad\Rightarrow\;$ series converges for all x.

(continued)

3) $\sum_1^\infty \frac{x^{2n+1}}{2^n n}$ ratio $= \frac{|x^{2n+3}/(2^{n+1}(n+1))|}{|x^{2n+1}/(2^n n)|} = |x|^2 \cdot \frac{n}{2(n+1)}$.

Taking the limit: $\lim_{n\to\infty} |x|^2 \cdot \frac{n}{2(n+1)} = \frac{x^2}{2}$

$\Rightarrow$ series converges if $\frac{x^2}{2} < 1$ i.e. for $|x| < \sqrt{2}$

Alternating series test (Do if time –which there never is)
If a_n alternates sign and $|a_n| \downarrow 0$ then $\sum a_n$ converges.

Example: Alternating harmonic series $\sum_1^\infty \frac{(-1)^n}{n} = \ln 2$

I.e. the series is conditionally convergent. (see Simmons bottom page 457 through example 3.)