18.01A Topic 14: Introduction to probability, discrete random variables.
Read: SN: P section 1

Read the supplementary notes

Repeatable experiments: Probability deals with repeatable experiments such as flipping a coin, rolling a die or measuring a distance. Gambling, polling and measuring are typical places where probability is used.

Discrete Random Variables Suppose the number of possible outcomes is finite.
Outcomes: $\{x_1, x_2, \ldots, x_n\}$.
Probabilities: $P(x_i)$ = probability of outcome x_i.

Example: Roll 1 die.
Outcomes = {1, 2, 3, 4, 5, 6}, with $P(j) = 1/6$ for $j = 1, \ldots, 6$.
E.g. if we roll a die many times we expect close to 1/6 of the outcomes to be a 2.

Example: Roll 2 die. There are two ways to describe the possible outcomes.
1. Ordered pairs = $\{(1,1), (1,2), \ldots, (6,6)\}$ with $P(i,j) = 1/36$ for any of the pairs.

2. Totals = $\{2, 3, 4, \ldots, 12\}$, here $P(2) = 1/36$, $P(7) = 6/36$ (you will need to compute all the probabilities for the pset).

Example: Toss a fair coin. Outcomes = {H, T}, with $P(H) = P(T) = 1/2$.

Example: Ask a random voter if they support candidate A.
Outcomes = { yes, no}, with $P(\text{yes}) = p$ and $P(\text{no}){=}1 - p$.

Note: In all the cases we carefully state how to run the repeatable experiment.

Terms
Sample space = set of possible outcomes = $\{x_1, x_2, \ldots, x_n\}$.
Probability function, $P(x_j)$ = probability of outcome x_j.
Trial = one run of the 'experiment'.
Independent: Two trials are independent if their outcomes have no effect on each other. E.g., repeated flips of a coin are independent.

Probability Laws:
Range law: i) $0 \leq P(x_j) \leq 1$, ii) $P(x_1) + \ldots + P(x_n) = 1$.
(This says the probability of an outcome is between 0 and 100% and the total probability of every possible outcome is 100%.

Addition law: $P(x_i \text{ or } x_j) = P(x_i) + P(x_j)$ (provided $x_i \neq x_j$).

Example: Roll two dice. Let A = 'the total is < 4' $\Rightarrow$ $A = \{2, 3\}$.
$P(A) = P(2 \text{ or } 3) = P(2) + P(3) = 1/36 + 2/36 = 1/12$.

Let B = 'the total is odd' = $\{3, 5, 7, 9, 11\}$.
$\Rightarrow$ $P(B) = P(3){+}P(5){+}P(7){+}P(9){+}P(11) = 2/36{+}4/36{+}6/36{+}4/36{+}2/36 = 1/2$.

Multiplication law: If two trials are independend the $P(x_i \text{ then } x_j) = P(x_i) \cdot P(x_j)$.

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Example: Two tosses of a coin. $P(\text{HH}) = 1/4 = P(\text{HT}) = P(\text{TH})) = P(\text{TT})$.

Example: Toss a fair die 3 times, what is the probablility of getting and odd number each time.
Let $A = \{1, 3, 5\}$. On any one toss $P(A) = 1/2$. Since repeated tosses are independent $P(A \text{ then } A \text{ then } A) = P(A) \cdot P(A) \cdot P(A) = 1/8$.

A **Finite Random Variable** X consists of
i) A list x_1, x_2 …, x_n of values X can take.
ii) A probability function $P(x_j)$.

Remark: The new requirement is that the outcomes have to be numbers.

Examples: 1. Roll a die, X = number of spots.
2. Roll a die, Y = (number of spots)^2.

Expectation (or **mean** or **expected value**) of the finite random variable X:

$$E(X) = x_1\, P(x_1) + \ldots + x_n\, P(x_n) = \sum_{i=1}^{n} x_i\, P(x_i).$$

Example: Roll a die, let X = number of spots.
$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5.$
Interpretation:
1. The expected value is the average over a lot of trials. E.g., roll a die, if I pay you \$1 per spot then over a lot of rolls you would average \$3.5 per roll.

Note, you would never be paid \$3.5 on any one turn, it is the expected average over many turns.

2. Expectation is a weighted average like center of mass.

Concept question: How much would you be willing to pay to roll the die?

Example: Roll a die, you win \$5 if it's a 1 and lose \$2 if it's not. Let X = your win or loss $\Rightarrow$ $P(X = 5) = 1/6$ and $P(X = -2) = 5/6$.
$E(X) = 5 \cdot \frac{1}{6} - 2\frac{5}{6} = -\frac{5}{6}.$
Concept question: Is the above bet a good one?

Infinite discrete random variable
i) X takes values x_1, x_2, ….
ii) Probability function $\sum_{1}^{\infty} P(x_j) = 1$.

(continued)

Example: (From supplementary notes) A trial consists of tossing a fair coin until it comes up heads. Let X = number of tosses $\Rightarrow$
Total probability $= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$.

n	1	2	3	4	...
toss pattern	H	TH	TTH	TTH	...
$P(n)$	1/2	1/4	1/8	1/16	...

Question: If I paid you \$$n$ for a trial of length n, what would you pay for a turn?
answer: Expectation $E(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \ldots = 2$. (The supplementary notes give a nice method for finding this sum.)

Poisson Random Variable with parameter m
i) X takes values 0, 1, 2, 3, ...
ii) $P(k) = \mathrm{e}^{-m}\frac{m^k}{k!}$. (The factor e^{-m} is chosen to give total probability 1.)
iii) Poisson random variables model events that occur *sparsely* (i.e. with low prob.).

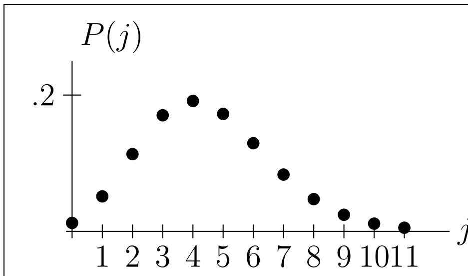
Examples: 1. Defects in manufacturing.
2. Errors in data transmission.
3. Number of cars in an hour at a rural tollbooth.
4. Number of chocolate chips in a cookie.

Theorem: $E(X) = m$.

Proof: $$E(X) = \sum_{k=0}^{\infty} k\mathrm{e}^{-m}\frac{m^k}{k!} = \mathrm{e}^{-m}(m + \frac{m^2}{1!} + \frac{m^3}{2!} + \ldots)$$
$$= m\mathrm{e}^{-m}(1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots) = m\mathrm{e}^{-m}\mathrm{e}^{m} = m.$$

Example: A manufacturer of widgets knows that 1/200 will be defective. What is the probability that a box of 144 contains no defective widgets?
answer: Since being defective is rare we can model the number of defective widgets in a box as a Poisson R.V. X with mean $m = 144/200$.
We need to find $P(X = 0)$:
$P(X = 0) = \mathrm{e}^{-m}\frac{m^0}{0!} = \mathrm{e}^{-144/200} = \underline{.487}$.
What is the probability that more than 2 are defective?
answer: $P(X > 2) =$
$1 - P(X = 0, 1, 2) = 1 - \mathrm{e}^{-m}(1 + m + \frac{m^2}{2!}) = .037$.

Histograms
If we ran many trials and made a histogram of the percentage of each outcome, the result should look like the graph of $P(x_j)$ vs. x_j.
Or we make a histogram of the count for each outcome.
The histograms at right give examples.



Histogram of percentages (Poisson distr. with $\mu = 4.5$)

90's: xx
80's: xxx
70's: xxxx
60's: xxx
50's: x

Histogram of counts of test scores