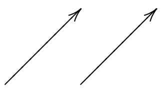
18.02A Topic 17: Vectors, dot product.
Read: TB: 17.3, 18.1, 18.2

Vectors

Two views: First the geometric and then the analytic.

Geometric view

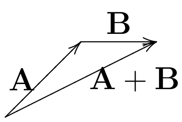
Vector = length and direction: (Discuss scaling, scalars)



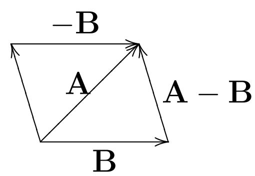
(same vector)

Length: denoted $|\mathbf{A}|$, also called magnitude or norm

Addition: (head to tail)



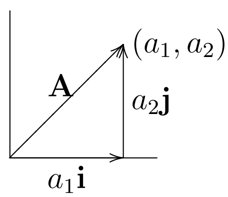
Subtraction: either tail to tail or $\mathbf{A}+(-\mathbf{B})$

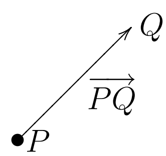


Analytic or algebraic view

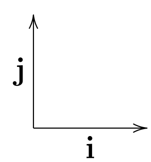
Place the tail of $\mathbf{A}$ at the origin $\Rightarrow$ the coordinates of the head determine $\mathbf{A}$:

$\mathbf{A} = \langle a_1,\, a_2\rangle = a_1\mathbf{i} + a_2\mathbf{j}$.





You've seen the vectors $\mathbf{i}$ and $\mathbf{j}$ in physics. They have coordinates $\mathbf{i} = \langle 1,\, 0\rangle$, $\mathbf{j} = \langle 0,\, 1\rangle$



Notation

1, (a_1, a_2) indicate as point in the plane.
2. $\langle a_1,\, a_2\rangle$ indicates the vector from the origin to the point (a_1, a_2). Of course, this vector can be translated anywhere and $\langle a_1,\, a_2\rangle = a_1\,\mathbf{i} + a_2\,\mathbf{j}$.
3. $\overrightarrow{\mathbf{P}} = \overrightarrow{\mathbf{OP}}$ is the vector from the origin to P.
4. In print we will often drop the arrow and just use the bold face to indicate a vector, i.e. $\mathbf{P} \equiv \overrightarrow{\mathbf{P}}$.

Discuss scaling, scalars
Length: $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}$
Addition: $(a_1\mathbf{i} + a_2\mathbf{j}) + (b_1\mathbf{i} + b_2\mathbf{j}) = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$
$\Leftrightarrow\ \langle a_1,\, a_2\rangle + \langle b_1,\, b_2\rangle = \langle a_1 + b_1,\, a_2 + b_2\rangle$
Discuss $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}$ –geom. and anal

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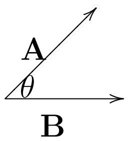
Dot product (scalar product)

Geometric definition:

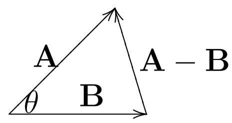
$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta$



(algebraic view)

$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2$ (Hard to get geometrically)

proof: Law of cosines: (won't do in class)



$|\mathbf{A}-\mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta$

$\Rightarrow\ (a_1^2+a_2^2) + (b_1^2+b_2^2) - ((a_1-b_1)^2 + (a_2-b_2)^2)$
$= 2|\mathbf{A}||\mathbf{B}|\cos\theta$

$\Rightarrow\ a_1b_1 + a_2b_2 = |\mathbf{A}||\mathbf{B}|\cos\theta.$ QED

Algebraic law: $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$.

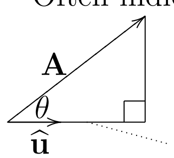
Follows from the algebraic view of dot product.

Unit vectors

Special vectors: $\mathbf{i}$ and $\mathbf{j}$. Note $\mathbf{i}\cdot\mathbf{i} = 1 = \mathbf{j}\cdot\mathbf{j}$ and $\mathbf{i}\cdot\mathbf{j} = 0$.

Unit vector: $\mathbf{u}$: $|\mathbf{u}| = 1$. Often indicate by $\widehat{\mathbf{u}}$.

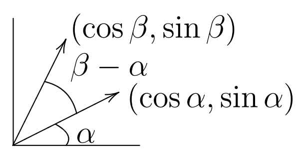
$\mathbf{A}\cdot\mathbf{u} = |\mathbf{A}|\cos\theta$
$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2$
$\mathbf{A}\perp\mathbf{B} \Leftrightarrow \mathbf{A}\cdot\mathbf{B} = 0$



$|\mathbf{A}|\cos\theta =$ component of $\mathbf{A}$ indirection of $\widehat{\mathbf{u}}$

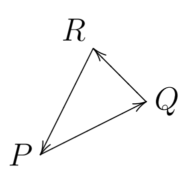
Components or projection:
The component of $\mathbf{A}$ in the direction of $\widehat{\mathbf{u}}$ is $\mathbf{A}\cdot\widehat{\mathbf{u}}$
For a non-unit vector: the component of $\mathbf{A}$ in the direction of $\mathbf{B}$ is the component of $\mathbf{A}$ in the direction of $\widehat{\mathbf{u}} = \frac{\mathbf{B}}{|\mathbf{B}|}$.

Trig identity $\cos(\beta-\alpha) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Unit vectors: $\mathbf{u} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$, $\mathbf{v} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$
Angle between them is $\theta = \beta - \alpha$
Geometric: $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = \cos\theta = \cos(\beta-\alpha)$
Analytic: $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.



Example: $P = (-5,0)$, $Q = (1,3)\ \Rightarrow\ \overrightarrow{\mathbf{PQ}} = 6\mathbf{i} + 3\mathbf{j} = \langle 6,\,3\rangle$.

Example: Show $\overrightarrow{\mathbf{PQ}} + \overrightarrow{\mathbf{QR}} + \overrightarrow{\mathbf{QP}} = 0$



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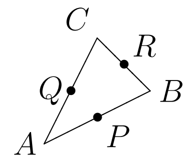
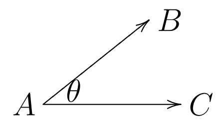
Example: Find 2 unit vectors parallel to $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.
$|\mathbf{v}| = 5$: $u_1 = \frac{3}{4}\mathbf{i} - \frac{4}{5}\mathbf{j}$, $\mathbf{u_2} = -\mathbf{u_1}$.

Example: Show the sum of the medians of a triangle = 0.



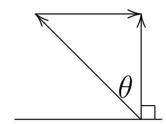
Median of $\overline{AB} = P = \frac{1}{2}(\mathbf{A} + \mathbf{B}) \Rightarrow \mathbf{CP} = \frac{1}{2}(\mathbf{B} + \mathbf{A}) - \mathbf{C}$.
Likewise: $\overrightarrow{\mathbf{BQ}} = \frac{1}{2}(\mathbf{A} + \mathbf{C}) - \mathbf{B}$, $\overrightarrow{\mathbf{AR}} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) - \mathbf{A}$.
$\Rightarrow$ Sum of medians $= \overrightarrow{\mathbf{CP}} + \overrightarrow{\mathbf{BQ}} + \overrightarrow{\mathbf{AR}} = 0$.

Example: Let $A = (1, 2)$, $B = (2, 3)$ and $C = (2, -1)$. Find the cosine of $\angle BAC$.



Let θ be the angle $\Rightarrow \cos\theta = \dfrac{\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{AC}}}{|\mathbf{AB}|\,|\mathbf{AC}|}$.
$\overrightarrow{\mathbf{AB}} = \langle 1,\ 1\rangle$, $\overrightarrow{\mathbf{AC}} = \langle 1, -3\rangle$
$\Rightarrow \cos\theta = \dfrac{1-3}{\sqrt{2}\,\sqrt{10}} = -\dfrac{2}{\sqrt{20}} = \dfrac{1}{\sqrt{5}}$.

Example: Velocities are vectors
A river flows at 3mph and a rower rows at 6mph. What heading should he use to get straight across a river?



Need $\sin\theta = \frac{3}{6} \Rightarrow \theta\pi/6$
Answer: Head at angle of $\pi/6$ radians upstream from straigh across.

Same question with river=2 mph, row=$2\sqrt{2}$ mph:
$\Rightarrow \sin\theta = \frac{2}{2\sqrt{2}} \Rightarrow \theta = \pi/4$.

Same question with river=6 mph, row=3 mph:
$\Rightarrow \sin\theta = \frac{6}{3} \Rightarrow$ No such θ

Three dimensions
Exactly the same except third coordinate: $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$

Example: Show $A = (4, 3, 6)$, $B = (-2, 0, 8)$, $C = (1, 5, 0)$ are the vertices of a right triangle.

Two legs of the triangle are $\overrightarrow{\mathbf{AC}} = \langle -3,\ 2,\ -6\rangle$
and $\overrightarrow{\mathbf{AB}} = \langle -6,\ -3,\ 2\rangle$
$\Rightarrow \overrightarrow{\mathbf{AC}} \cdot \overrightarrow{\mathbf{AB}} = 18 - 6 - 12 = 0 \Rightarrow$ orthogonal.