18.02A Topic 18: Determinants, cross-product.
Read: SN: D, TB: 18.3

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \text{“diag” - “antidiag”}$$

Example: $\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix} = 7.$

Row operations: scale row = scale det: $\begin{vmatrix} 12 & 10 \\ 1 & 2 \end{vmatrix} = 14$

swap rows = change sign: $\begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix} = -7.$

Algebraic facts for $|A|$

1. 1 row or column all 0's $\Rightarrow$ $|A| = 0$.
2. Multiply row (or column) by c $\Rightarrow$ det multiplied by c.
3. Interchange two rows (or columns) $\Rightarrow$ det $= -|A|$
4. Add multiple of one row (or column) to another $\Rightarrow$ det unchanged.

More examples

Scale row 2 by 2: $\begin{vmatrix} 6 & 5 \\ 2 & 4 \end{vmatrix} = 14$ Add row 1 to row 2: $\begin{vmatrix} 6 & 5 \\ 7 & 7 \end{vmatrix} = 7$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (\text{add } R_1 \text{ to } R_3) \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{vmatrix} = (\text{add } -2R_2 \text{ to } R_3) \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 5 & 6 & 7 \end{vmatrix} = 0$$

Laplace formula for determinants of 3×3 matrices and bigger.

$a_{i,j}$ notation: i=row, j=column: $A = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$

i, j minor = det after removing i^{th}row and j^{th}column (i.e. the row and col. with $a_{i,j}$.

i, j cofactor $= (-1)^{i+j}(i,\ j \text{ minor})$. I.e. sign $= \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

$|A| = \sum$ entry$\times$cofactor

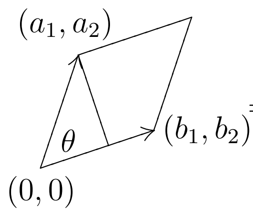
Example: $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} (\text{along top row}) = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 0$$

(continued)

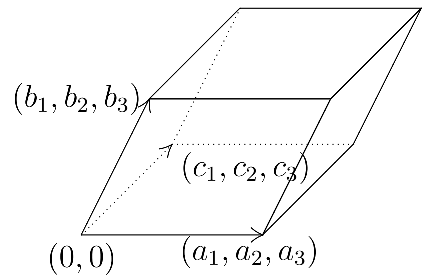
Example: $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 0 & 7 \\ 8 & 0 & 9 \end{vmatrix}$ Use second column: $\det = -2 \cdot \begin{vmatrix} 5 & 7 \\ 8 & 9 \end{vmatrix} + 0 \cdot * - 0 \cdot * = 22$

Geometry

$\pm \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ = area of

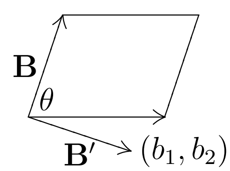


$= |A||B|\sin\theta$



$\pm \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = volume parallelpiped

Example: Rotate B by $\pi/2$ clockwise $\Rightarrow$ $B' = (b_2, -b_1)$
Area $= |A||B|\sin\theta = |A||B|\cos(\pi/2 - \theta) = A \cdot B' = a_1 b_2 - a_2 b_1$



Example: Volume of parallelpiped with vertices $(0,0,0)$, $(1,2,3)$, $(5,0,7)$, $8,0,9)$ $= 22$.

Cross product

Geometric definition of $\mathbf{A} \times \mathbf{B}$

length = area $= |A||B|\sin\theta$

direction = right hand rule (draw picture)

Algebraic facts:

$\mathbf{A} \times \mathbf{A} = \mathbf{0}$, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ (not obvious)

Non-associative: $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ (example in a moment)

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

(i.e. cycle: $\mathbf{i} \to \mathbf{j} \to \mathbf{k}$) (in diagram loop **k** back to **i**, also add pictures)

Examples: $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = -\mathbf{i}$ but $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = 0$.

$(2\mathbf{i} + 3\mathbf{j}) \times (3\mathbf{i} - 2\mathbf{j}) = -13\mathbf{k}$ (Picture –geometry, algebra –distribute)

Determinants for cross product

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Example: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \mathbf{k} = -13\mathbf{k}$

DON'T FORGET THE GEOMETRY -it will be used to solve problems.