18.02A Topic 19: Matrices, inverses.
Read: SN: M.1, M.2

Introduction Making Halloween bags (contents of bags):

	Kids	Teens	Adults
Lollipops	2	1	0
Snickers	1	2	2
Candy corn	1	2	2

Output = no. bags = $\begin{pmatrix} K \\ T \\ A \end{pmatrix} = \overrightarrow{\mathbf{x}}$; Input = amount candy = $\begin{pmatrix} L \\ S \\ C \end{pmatrix} = \overrightarrow{\mathbf{c}}$

Given $\overrightarrow{\mathbf{x}}$ can find $\overrightarrow{\mathbf{c}}$, i.e. $L = 2 \cdot K + 1 \cdot T + 0 \cdot A$ etc.
Given $\overrightarrow{\mathbf{c}}$ can find $\overrightarrow{\mathbf{x}}$ by solving simultaneous equations.

Matrix notation

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} K \\ T \\ A \end{pmatrix} = \begin{pmatrix} L \\ S \\ C \end{pmatrix} = \begin{pmatrix} (2,1,0) \cdot \overrightarrow{\mathbf{x}} \\ (1,2,1) \cdot \overrightarrow{\mathbf{x}} \\ (1,2,2) \cdot \overrightarrow{\mathbf{x}} \end{pmatrix} \quad \text{Or simply: } A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{c}}$$

The matrix A is the **coefficient matrix** and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is called a **column vector**.

Demonstrate matrix multiplication: requires two hands. $\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 21 \\ 7 \end{pmatrix}$.

Sizes $n \times m = n$ rows and m columns

Examples:

$\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$: 2×2 $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$: 3×1 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: 3×3 $(1, 2, 3)$: 3×1, (3): 1×1

Matrix multiplication extends to matrix times matrix.
$A \cdot B$: i, j entry $= <i^{\text{th}}$ row of $A > \cdot < j^{\text{th}}$ column of $B >$
$\Rightarrow$ must have #columns of A = #rows of B.

Examples:

$$\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 16 & 38 & 60 \\ 5 & 11 & 17 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 9 \\ 22 & 23 \\ 36 & 37 \end{pmatrix}$$

$\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ not valid expression.

(continued)

Algebraic laws

1. Identity: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot A = A$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot A = A$, etc.

In general, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc.

If the size is clear from context, we just write I: $I \cdot A = A \cdot I = A$

2. Addition: $A + B$ add termwise (must be the same size)
3. Scalar multiplication: cA multiply scalar times matrix termwise.
4. Distributive law: $(A + B) \cdot C = A \cdot C + B \cdot C$ (easy to show)
5. Associative law: $(AB)C = A(BC)$ (challlenging to show)
6. **NOT** commutative: $AB \neq BA$. (Doesn't even make sense –see examples above.)

Inverses

For scalars: $ax = c \Rightarrow x = a^{-1}c$
What is a^{-1}? Answer: the number such that $a^{-1}a = 1$.
Matrices are the same: A^{-1} is the matrix (if it exists) such that $A^{-1}A = I$
$\Rightarrow$ to solve $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{c}}$ for $\overrightarrow{\mathbf{x}}$
Multiply by A^{-1} on left: $\Rightarrow A^{-1}A\overrightarrow{\mathbf{x}} = A^{-1}\overrightarrow{\mathbf{c}}$
$\Rightarrow I\overrightarrow{\mathbf{x}} = A^{-1}\overrightarrow{\mathbf{c}}$
$\Rightarrow \overrightarrow{\mathbf{x}} = A^{-1}\overrightarrow{\mathbf{c}}$

Because matrix multiplication is not commutative it is important to multiply on the correct side. E.g. in the example above $A\overrightarrow{\mathbf{x}}A^{-1}$ would make no sense.

When do we have inverses?

For scalars: a^{-1} exists if $a \neq 0$.

For matrices: A must be square and $|A| \neq 0$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example:

$\begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{7} \begin{pmatrix} 2 & -5 \\ -1 & 6 \end{pmatrix}$ (verify this by multiplication)

(continued)

Finding inverses via the adjoint method

Start with A

$\rightarrow$ matrix of minors

$\rightarrow$ matrix of cofactors (use $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ to change signs)

$\rightarrow$ Adjoint (transpose rows with columns)

$\rightarrow$ Divide by $|A|$

Example: $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

$|A| = -4$ (Do this first, to see if the inverse even exists)

Minors: $\begin{pmatrix} -2 & 0 & 1 \\ 2 & 4 & 3 \\ 2 & 4 & 1 \end{pmatrix}$

Cofactors: $\begin{pmatrix} -2 & 0 & 1 \\ -2 & 4 & -3 \\ 2 & -4 & 1 \end{pmatrix}$

Adjoint: $\begin{pmatrix} -2 & -2 & 2 \\ 0 & 4 & -4 \\ 1 & -3 & 1 \end{pmatrix}$

$A^{-1} = -\frac{1}{4} \begin{pmatrix} -2 & -2 & 2 \\ 0 & 4 & -4 \\ 1 & -3 & 1 \end{pmatrix}$

Check this by multiplying $A \cdot A^{-1}$.

NOTE: This works for 2×2 case also.