18.01A Topic 2: Higher order approximations, Taylor series, Mean-value theorem.
Read: Orloff class notes on this topic, TB: 2.6 to middle p. 77, SN: MVT.

Higher order approximations and Taylor series

Why stop at quadratic approximations?
Going to cubic approximation near a:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.$$

Here 3! is read as '3 factorial' and means $3 \cdot 2 \cdot 1$. Likewise $4! = 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1$ etc.
By convention $0! = 1$.

A fourth order approximation near a is given by

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4.$$

In general there is the **Taylor series** for $f(x)$ near a which keeps an infinite number of terms. (To emphasize the pattern we keep the 0! and the 1!.)

$$f(x) \approx \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

Examples: (all taking $a = 0$) These are important, you should learn them.

1. Exponential function: $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

We can use this to approximate $\mathrm{e} = \mathrm{e}^1 \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.71$.

Rule of thumb: error $\approx$ next term.

In this example, error $\approx 1/5! \approx .01$.

To many decimal places $\mathrm{e} = 2.7182818284590451$.

2. Geometric series: $1 + x + x^2 + \ldots = \frac{1}{1-x}$ valid for $-1 < x < 1$.

We can work backwards to see the geometric series is just the Taylor series for $\frac{1}{1-x}$.

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(continued)

5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ (Trick to get this in a moment.)

Tricks:

1. Algebra:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \Rightarrow e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$$

2. Differentiation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \Rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

3. Antidifferentiation (also called integration)

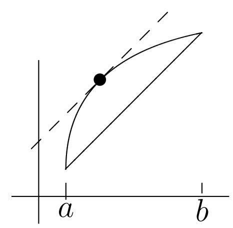
$$\frac{d\ln(1+x)}{dx} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \Rightarrow \ln(1+x) = x - x^2 + \frac{x^3}{3} - \ldots$$

Mean-Value Theorem (MVT)

We often want to know how big the error can be when we make an approximation. One theorem that helps is the mean-value theorem. We won't do any theoretical work with the MVT, but you should know that it plays a big role in proving many of the main theorems.

Statement 1 (slope form):

If $f(x)$ is differentiable on $[a, b]$ then there is a number c with $a < c < b$ and $\frac{f(b) - f(a)}{b - a} = f'(c)$.



Statement 2 (analytic form): If f differentiable then is a c between a and x such that $f(x) = f(a) + f'(c)(x - a)$.

proof of statement 2: Simple algebra shows this is equivalent to statement 1.

Examples:

1. Show $e^x > 1 + x$ for $x > 0$.

<u>**answer:**</u> Use the analytic form with $f(x) = e^x$ and $a = 0$:

Since $c > 0$ we know $e^c > 1 \Rightarrow f(x) = 1 + f'(c)x = 1 + e^c x > 1 + x$.

With more algebra can show this holds for $x < 0$ also:

$x < 0 \Rightarrow c < 0 \Rightarrow e^c < 1 \Rightarrow e^c x > x$ (less negative) $\Rightarrow 1 + e^c x > 1 + x$.

2. Show if $f'(x) > 0$ on $[a, b]$ then f is increasing.

<u>**answer:**</u> Suppose $a < x_1 < x_2 < b$. We need to show $f(x_1) < f(x_2)$.

MVT (with x_1 in place of a) $\Rightarrow f(x_2) = f(x_1) + f'(c)(x_2 - x_1)$ for some $x_1 < c < x_2$.

Since $f'(c)$ and $x_2 - x_1$ are both positive this shows $f(x_2) > f(x_1)$. ■

(continued)

3. Show if $f'(x) = 0$ on $[a, b]$ then f is constant.
answer: MVT $\Rightarrow$ $f(x) = f(a) + f'(c)(x-a) = f(a) + 0 \cdot (x-a) = f(a)$. ∎

4. Show if $f'(x) < 0 \Rightarrow f$ decreasing.
answer: Same as example 2.

5. Find c (as in the MVT) for $f(x) = x^3$ on $[0, 1]$.
answer: $x^3 = 0 + 3c^2 x \Rightarrow c = x/\sqrt{3}$.

6. Show $\ln x \leq x - 1$ for $x > 0$.
answer: Let $a = 1$. MVT $\Rightarrow \ln x = 0 + \frac{1}{c}(x-1)$.
We examine the cases i) $x < 1$ and ii) $x > 1$ separately.
i) $x > 1 \Rightarrow c > 1 \Rightarrow 1/c < 1$ and $x - 1 > 0$.
$\Rightarrow \ln x = \frac{1}{c}(x-1) < x - 1$.
ii) $x < 1 \Rightarrow c < 1 \Rightarrow 1/c > 1$ and $x - 1 < 0$.
$\Rightarrow \ln x = \frac{1}{c}(x-1) < x - 1$. ∎

7. Problem in notes 2G-4: A polynomial $p(x)$ of degree n has at most n roots (but it may have fewer, e.g. $x^2 + 1$). Show that if $p(x)$ has n distinct roots then $p'(x)$ has $n - 1$ distinct roots.
answer: Apply Rolle's theorem between each pair of roots

(These proofs are for entertainment only.)

Speed proof of statement 1:
Suppose you drive down the highway. If $f(t)$ is your position at time t then $f(b) - f(a)$ is the distance traveled in the time interval $[a, b]$ and $\frac{f(b)-f(a)}{b-a}$ is the average speed. If your average is 60 mph then you can't have gone under (or over) 60 the whole way. So at some point you must have been going 60, i.e. ave = $f'(c)$ for some c.

Analytic proof of statement 1:
First we need to state **Rolle's Theorem**:
If f differentiable on $[a, b]$ and $f(a) = f(b) = 0$ then there is a c between a and b such that $f'(c) = 0$. (proof: speed argument –start and end at same place means never left or else turned around).

Proof of MVT: Tilt Rolle's picture, i.e.
$g(x) = f(x) - (f(a) + \frac{f(b)-f(a)}{b-a}(x-a))$.
Since $g(a) = g(b) = 0$, Rolle's Theorem $\Rightarrow g'(c) = 0$.
$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$.