18.02A Topic 20: Square matrices/systems, Cramer's rule, planes.
Read: SN: M.3, M.4

Square Systems
We look at 3×3 (and 2×2) but this applies to all dimensions.

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \leftrightarrow A \cdot \overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{d}}$$

The goal is to solve for $\overrightarrow{\mathbf{X}}$ when A and $\overrightarrow{\mathbf{d}}$ are known.

First we study the case where $|A| \neq 0$, i.e. where A^{-1} exists.
Working carefully: $A \cdot \overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{d}}$
$\Rightarrow\ A^{-1}(A \cdot \overrightarrow{\mathbf{X}}) = A^{-1} \cdot \overrightarrow{\mathbf{d}}$
$\Rightarrow\ (A^{-1}A)\overrightarrow{\mathbf{X}} = A^{-1} \cdot \overrightarrow{\mathbf{d}}$
$\Rightarrow\ I \cdot \overrightarrow{\mathbf{X}} = A^{-1} \cdot \overrightarrow{\mathbf{d}}$
$\Rightarrow\ \overrightarrow{\mathbf{X}} = A^{-1} \cdot \overrightarrow{\mathbf{d}}$

Conclusion: If $\lvert A\rvert \neq 0$ then there is exactly one solution: $\overrightarrow{\mathbf{X}} = A^{-1} \cdot \overrightarrow{\mathbf{d}}$.

Example: Using the example from last time:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}, \quad |A| = -4, \quad A^{-1} = -\frac{1}{4} \begin{pmatrix} -2 & -2 & 2 \\ 0 & 4 & -4 \\ 1 & -3 & 1 \end{pmatrix}$$

1. $\overrightarrow{\mathbf{X}} = A^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} -2 & -2 & 2 \\ 0 & 4 & -4 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/4 \end{pmatrix}$ (check this answer)

2. Solve $A \cdot \overrightarrow{\mathbf{X}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ \Rightarrow\ \overrightarrow{\mathbf{X}} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(This one is easy to solve –but note the analysis above guarantees this is the only solution.)

Now we look at the case $|A| = 0$ (i.e. where A^{-1} doesn't exist).
1. $A \cdot \overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{0}}$ has infinitely many solutions. (**Homogeneous case**)
2. If $\overrightarrow{\mathbf{d}} \neq \overrightarrow{\mathbf{0}}$ then $A \cdot \overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{d}}$ has either 0 or many solutions depending on $\overrightarrow{\mathbf{d}}$.

Example: (1×1 case)

$7x = 5$	one solution	$7x = 0$	one solution
$0x = 5$	no solutions	$0x = 0$	infinitely many solution

(continued)

Here's the reasoning in the 2×2 case.

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \Rightarrow \begin{array}{rcrcl} x & + & 2y & = & d_1 \\ 3x & + & 6y & = & d_2 \end{array}$

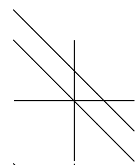
Each of these equations is the equation of a line.

Geometry

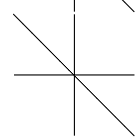
Geometrically solving systems of equations means finding the intersection of these two lines. $|A| = 0$ means the two rows are multiples of each other, i.e. the two lines are parallel.

Two possibilities:

1. The lines are different $\Rightarrow$ no solutions:

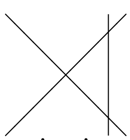


2. The lines are the same $\Rightarrow$ infinitely many solutions:

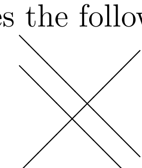


In the homogeneous case the lines are automatically the same (parallel and through the origin –see above picture).

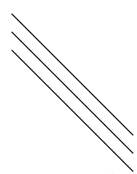
For the 3 by 3 case, there are more possibilities. Geometrically, solving means finding the intersection of three planes. If $|A| = 0$ then the 3 planes are all perpendicular to one plane (volume $= 0 \Rightarrow$ rows of A are all in a plane $\Rightarrow$ normals all in this plane). A head on view gives the following possibilities:



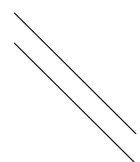
pairwise transverse
no solutions



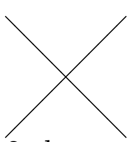
2 parallel,
1 transverse
no solutions



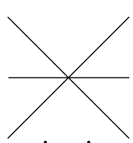
3 parallel
no solutions



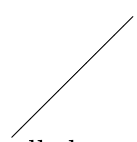
2 the same, 1 parallel
no solutions



2 the same,
1 transverse
many solutions



pairwise transverse
many solutions



all the same
many solutions

Summary

	$\lvert A\rvert \neq 0$	$\lvert A\rvert = 0$
Homogeneous	1 solution: $\overrightarrow{\mathbf{X}} = 0$	many solutions
Inhomogeneous	1 solution: $\overrightarrow{\mathbf{X}} = A^{-1}\overrightarrow{\mathbf{d}}$	depends on $\overrightarrow{\mathbf{d}}$

NOTE: For the 1 solution cases –*no matter* how it's found the solution is unique.

(continued)

Example: Solve $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{\mathbf{0}}$

det $= 0 \;\Rightarrow\;$ many solutions (all vectors perpendicular to all 3 rows).

Cross product: $\langle 1,\, 2,\, 3\rangle \times \langle 4,\, 5,\, 6\rangle = \langle -3,\, 6,\, -3\rangle \;\Rightarrow\;$ solutions $= c \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$

Example: For what c does the following have a non-zero solution?

$$\begin{array}{ccccccc} 2x & + & & + & cz & = & 0 \\ x & - & y & + & 2z & = & 0 \\ x & - & 2y & + & 2z & = & 0 \end{array}$$

In matrix form: $\begin{pmatrix} 2 & 0 & c \\ 1 & -1 & 2 \\ 1 & -2 & 2 \end{pmatrix} \vec{\mathbf{X}} = \vec{\mathbf{0}} \;\Rightarrow\;$ want $\begin{vmatrix} 2 & 0 & c \\ 1 & -1 & 2 \\ 1 & -2 & 2 \end{vmatrix} = 0$

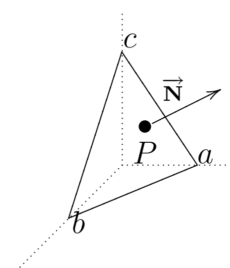
$\Rightarrow\; 4 - c = 0 \;\Rightarrow\; c = 4$

In this case, $\vec{\mathbf{x_0}} = \langle 2,\, 0,\, 4\rangle \times \langle 1,\, -1,\, 2\rangle = \langle 4,\, 0,\, -2\rangle$ is a solution (as is $a \cdot \vec{\mathbf{x_0}}$ for any a).

Example: For what d_1 and d_2 does the following have a solution?

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \;\Rightarrow\; x = d_1 \text{ and } 0 = d_2 \quad \Rightarrow \begin{cases} \text{if} \quad d_2 \neq 0 & \text{no solutions} \\ \text{if} \quad d_2 = 0 & \begin{pmatrix} d_1 \\ y \end{pmatrix} \text{ a solution for any } y \end{cases}$$

Planes



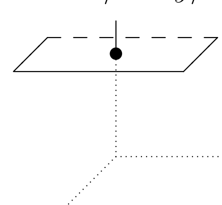
$\vec{\mathbf{N}} =$ normal to plane at $\vec{\mathbf{P}}$

$\vec{\mathbf{X}} = \langle x,\, y,\, z\rangle =$ any point in plane

$\overrightarrow{\mathbf{PX}} \cdot \vec{\mathbf{N}} \Leftrightarrow (\vec{\mathbf{X}} - \vec{\mathbf{P}}) \cdot \vec{\mathbf{N}} = 0 \;\Rightarrow\; \vec{\mathbf{X}} \cdot \vec{\mathbf{N}} = \vec{\mathbf{P}} \cdot \vec{\mathbf{N}}$

Example: $\vec{\mathbf{N}} = \langle -a,\, b,\, 0\rangle \times \langle -a,\, 0,\, c\rangle = \langle bc,\, ac,\, ab\rangle$; $\quad P = (a, 0, 0)$

$\Rightarrow\;$ Eq. of plane: $\vec{\mathbf{X}} \cdot \vec{\mathbf{N}} = \vec{\mathbf{P}}\vec{\mathbf{N}} \;\Rightarrow\; bcx + acy + abz = abc \Leftrightarrow x/a + y/b + z/c = 1$



Example:

Normal $= \vec{\mathbf{N}} = \widehat{\mathbf{k}}$; $\quad \mathbf{P} = \langle 0,\, 0,\, 3\rangle$

Eq. of plane: $\widehat{\mathbf{k}} \cdot \vec{\mathbf{X}} = 3 \Leftrightarrow z = 3$

System: (go through slowly)

$$\left.\begin{array}{rcl} \vec{\mathbf{A}} \cdot \vec{\mathbf{X}} & = & d_1 \\ \vec{\mathbf{B}} \cdot \vec{\mathbf{X}} & = & d_2 \\ \vec{\mathbf{C}} \cdot \vec{\mathbf{X}} & = & d_3 \end{array}\right. = \text{intersection of 3 planes} \to \text{usually a point.}$$

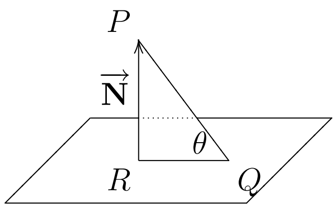
(continued)

Distances:

1. *Distance point to plane:*
Ingredients: i) A point P, ii) A plane with normal $\overrightarrow{\mathbf{N}}$ and point Q.

The distance from P to the plane is $d = |\overrightarrow{\mathbf{PQ}}|\cos\theta = \left|\overrightarrow{\mathbf{PQ}} \cdot \dfrac{\overrightarrow{\mathbf{N}}}{|\mathbf{N}|}\right|$.

Example: Let $P = (1,3,2)$. Find the distance from P to the plane $x + 2y = 3$.



Q = any point on the plane, we take $Q = (3,0,0)$.
$\mathbf{N}$ = normal to plane = $\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}}$.
R = point on plane closest to P (unknown).
Distance = $\left|\text{Proj}_{\mathbf{N}}\overrightarrow{\mathbf{PQ}}\right| = \left|\overrightarrow{\mathbf{PQ}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|}\right| = |\overrightarrow{\mathbf{PQ}}|\cos\theta$.
$\overrightarrow{\mathbf{PQ}} = 2\widehat{\mathbf{i}} - 3\widehat{\mathbf{j}} - 2\widehat{\mathbf{k}}, \;\Rightarrow\; d = |\overrightarrow{\mathbf{PQ}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|}| = \frac{3}{\sqrt{5}}$.

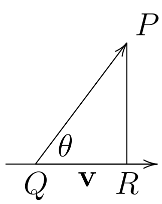
2. *Distance point to line:*
Ingredients: i) A point P, ii) A line with direction vector $\mathbf{v}$ and point Q.

The distance from P to the line is $d = |\mathbf{QP}|\sin\theta = \left|\overrightarrow{\mathbf{QP}} \times \dfrac{\mathbf{v}}{|\mathbf{v}|}\right|$.

An alternate formula using projection is

$$\overrightarrow{\mathbf{QR}} = \text{Proj}_{\mathbf{v}}\overrightarrow{\mathbf{QP}} = \left(\overrightarrow{\mathbf{QP}} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}, \qquad d = |\overrightarrow{\mathbf{RP}}| = |\overrightarrow{\mathbf{QP}} - \overrightarrow{\mathbf{QR}}|.$$

Example: Let $P = (1,3,2)$, find the distance from the point P to the line through $(1,0,0)$ and $(1,2,0)$.



Q = any point on the line, we take $Q = (1,0,0)$.
$\mathbf{v}$ = direction vector of line = $\langle 1,\,2,\,0\rangle - \langle 1,\,0,\,0\rangle = 2\widehat{\mathbf{j}}$.
R = point on line closest to P (unknown).
$\overrightarrow{\mathbf{QP}} = 3\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}$.

Method 1: cross product formula: $d = |\overrightarrow{\mathbf{QP}} \times \frac{\mathbf{v}}{|\mathbf{v}|}| = |(3\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}) \times \widehat{\mathbf{j}}| = 2$.

Method 2: projection formula: $\overrightarrow{\mathbf{QR}} = \left(\overrightarrow{\mathbf{QP}} \cdot \dfrac{\mathbf{v}}{|\mathbf{v}|}\right)\dfrac{\mathbf{v}}{|\mathbf{v}|} = 3\widehat{\mathbf{j}} \;\Rightarrow\; d = |\overrightarrow{\mathbf{QP}} - \overrightarrow{\mathbf{QR}}| = |2\widehat{\mathbf{k}}| = 2$.

3. *Distance between parallel planes:* Reduce to the distance from a point to a plane.

Example: Find the distance between the planes $x + 2y - z = 4$ and $x + 2y - z = 3$.
Both planes have normal $\mathbf{N} = \widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} - \widehat{\mathbf{k}}$ so they are parallel.
Take any point on the first plane, say, $P = (4,0,0)$.
Distance between planes = distance from P to second plane.
Choose $Q = (1,0,0)$ = point on second plane
$\Rightarrow\; d = |\overrightarrow{\mathbf{QP}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|}| = |3\widehat{\mathbf{i}} \cdot (\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} - \widehat{\mathbf{k}})|/\sqrt{6} = \sqrt{6}/2$.

4. *Distance between skew lines:* Put the lines in parallel planes.
Normal to planes = $\mathbf{N} = \mathbf{v_1} \times \mathbf{v_2}$, where $\mathbf{v_1}$ and $\mathbf{v_2}$ are the dir. vectors of the lines.