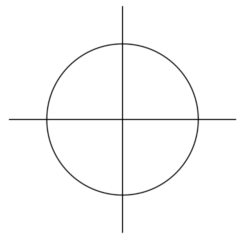
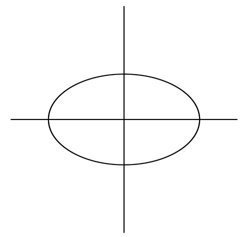
18.02A Topic 21: Parametric equations.
Read: TB: 17.1, 17.2 to middle of page 598, 18.4

Parametric Equations
Circles

$$\begin{aligned} x(t) &= a\cos t \\ y(t) &= a\sin t \end{aligned} \Leftrightarrow \text{circle } x^2+y^2=a^2$$

$$\begin{aligned} x(t) &= a\cos t \\ y(t) &= b\sin t \end{aligned} \Leftrightarrow \text{ellipse } \tfrac{x^2}{a^2}+\tfrac{y^2}{b^2}=1$$



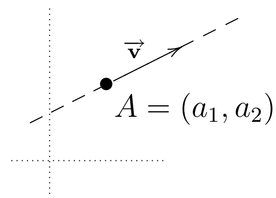


Lines

parametric form (t=param.) symmetric form

$$\vec{\mathbf{X}} = \vec{\mathbf{A}} + t\,\vec{\mathbf{v}}$$

$$\Leftrightarrow \quad \begin{pmatrix} x &=& a_1+v_1t \\ y &=& a_2+v_2t \end{pmatrix} \quad \Leftrightarrow \quad \frac{x-a_1}{v_1} = \frac{y-a_2}{v_2}$$



Example: Find the line through $(0,-2,1)$ and $(1,0,2)$
(Some students have trouble with the idea that $\vec{\mathbf{PQ}} = \vec{\mathbf{Q}} - \vec{\mathbf{P}}$)
$\vec{\mathbf{v}} = \vec{\mathbf{PQ}} = \langle 1,2,1\rangle$

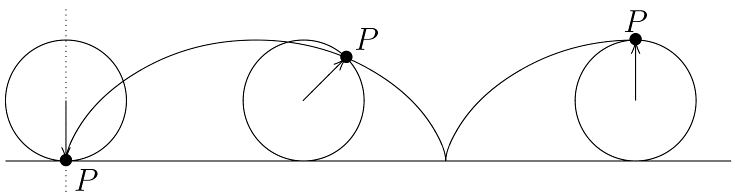
$$\Rightarrow \vec{\mathbf{X}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \langle t, -2+2t, 1+t\rangle = t\,\hat{\mathbf{i}} + (-2+2t)\,\hat{\mathbf{j}} + (1+t)\,\hat{\mathbf{k}}$$

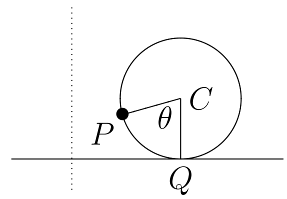
General parametric equations:

Notation: $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$ = **Position** vector

(continued)

Example: Cycloid (will use this often)
Roll a wheel (circle or radius a) along the x-axis and follow the trajectory of a point on the wheel –this is a cycloid.
Brachistochrone -Bernouilli, Tautochrone –Huygens





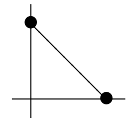
To find parametric equations we use vectors:
$\overrightarrow{\mathbf{r}}(\theta) = \overrightarrow{\mathbf{OQ}} + \overrightarrow{\mathbf{QC}} + \overrightarrow{\mathbf{CP}}$
$\overrightarrow{\mathbf{OQ}} = \langle a\theta, 0\rangle$ = amount rolled (a=radius)
$\overrightarrow{\mathbf{QC}} = \langle 0, a\rangle$
$\overrightarrow{\mathbf{CP}} = \langle -a\sin\theta, -a\cos\theta\rangle$

$$\begin{aligned} \Rightarrow \ \overrightarrow{\mathbf{r}}(\theta) &= \langle a\theta - a\sin\theta, a - a\cos\theta\rangle \\ &= a\langle \theta - \sin\theta, 1 - \cos\theta\rangle \\ &= a(\langle \theta - \sin\theta\rangle\,\widehat{\mathbf{i}} + \langle 1 - \cos\theta\rangle\,\widehat{\mathbf{j}}\rangle \end{aligned}$$

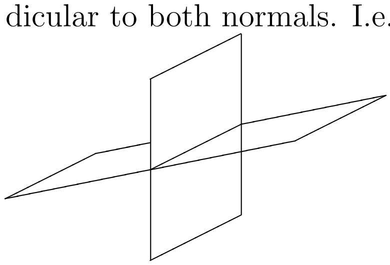
NOTE: symmetric form of equations is hard to write down

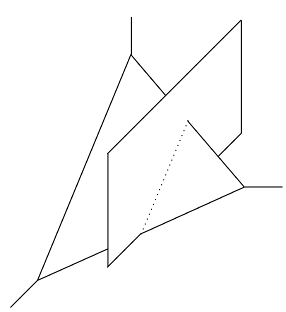
Example: Where symmetric form loses information
$x = a\cos^2 t,\ y = a\sin^2 t \ \Leftrightarrow\ x + y = a,\ x, y$ non-negative

Elimination loses information. I.e. the parametric equations show how the curve is traced out.



Intersection of two planes:
If $\overrightarrow{\mathbf{N}}_1$ and $\overrightarrow{\mathbf{N}}_2$ are the normals to the planes then the line of intersection is perpendicular to both normals. I.e. it has direction vector = $\overrightarrow{\mathbf{N}}_1 \times \overrightarrow{\mathbf{N}}_2$





Example: Find the intersection of the planes $x + y + z = 1$ and $y = 2$.
Normals: $\overrightarrow{\mathbf{N}}_1 = \langle 1, 1, 1\rangle$ and $\overrightarrow{\mathbf{N}}_2 = \langle 0, 1, 0\rangle$.
Direction vector: $\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{N}}_1 \times \overrightarrow{\mathbf{N}}_2 = \widehat{\mathbf{i}} - \widehat{\mathbf{k}} = \langle 1, 0, -1\rangle$.
One point of intersection: (by elimintation) $y = 2 \ \Rightarrow\ x + 2 + z = 3 \ \Rightarrow\ P = (1, 2, 0)$.
Answer: The intersection is the line $\langle 1, 2, 0\rangle + t\langle 1, -1, 0\rangle$.