18.02A Topic 21: Parametric equations.
Read: TB: 17.1, 17.2 to middle of page 598, 18.4

Parametric Equations

Circles

\[
\begin{align*}
  x(t) &= a \cos t \\
  y(t) &= a \sin t \\
  x(t) &= a \cos t \\
  y(t) &= b \sin t
\end{align*}
\]

\[\iff\] circle \( x^2 + y^2 = a^2 \)

Ellipse

\[
\begin{align*}
  x(t) &= a \cos t \\
  y(t) &= b \sin t
\end{align*}
\]

\[\iff\] ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

Lines

parametric form (t=param.) symmetric form

\[
\begin{align*}
  \vec{r} &= \vec{A} + t\vec{v} \\
  \begin{cases} 
    x &= a_1 + v_1 t \\
    y &= a_2 + v_2 t 
  \end{cases} & \iff \begin{cases} 
    x - a_1 &= y - a_2 \\
    v_1 &= v_2 
  \end{cases}
\end{align*}
\]

Example: Find the line through \((0, -2, 1)\) and \((1, 0, 2)\)

(Some students have trouble with the idea that \(\vec{PQ} = \vec{Q} - \vec{P}\))

\[
\vec{v} = \vec{PQ} = \langle 1, 2, 1 \rangle
\]

\[
\begin{align*}
  \vec{r} &= \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \langle t, -2 + 2t, 1 + t \rangle = t\hat{i} + (-2 + 2t)\hat{j} + (1 + t)\hat{k}
\end{align*}
\]

General parametric equations:

Notation: \(\vec{r}(t) = \langle x(t), y(t) \rangle = x(t)\hat{i} + y(t)\hat{j} = \text{Position vector}\)

(continued)
**Example:** Cycloid (will use this often)
Roll a wheel (circle or radius \(a\)) along the \(x\)-axis and follow the trajectory of a point on the wheel –this is a cycloid.
Brachistochrone -Bernouilli, Tautochrone –Huygens

![Cycloid diagram]

To find parametric equations we use vectors:
\[
\mathbf{r}(\theta) = \mathbf{OQ} + \mathbf{QC} + \mathbf{CP}
\]
\[
\mathbf{OQ} = \langle a\theta, 0 \rangle = \text{amount rolled (a=radius)}
\]
\[
\mathbf{QC} = \langle 0, a \rangle
\]
\[
\mathbf{CP} = \langle -a\sin\theta, -a\cos\theta \rangle
\]
\[
\Rightarrow \mathbf{r}(\theta) = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle
\]
\[
= a(\theta - \sin\theta, 1 - \cos\theta)
\]
\[
= a(\langle \theta - \sin\theta \rangle \mathbf{i} + \langle 1 - \cos\theta \rangle \mathbf{j})
\]

**NOTE:** symmetric form of equations is hard to write down

**Example:** Where symmetric form loses information
\[
x = a\cos^2 t, \ y = a\sin^2 t \Leftrightarrow x + y = a, \ x, y \text{ non-negative}
\]

Elimination loses information. I.e. the parametric equations show how the curve is traced out.

**Intersection of two planes:**

If \(\mathbf{N}_1\) and \(\mathbf{N}_2\) are the normals to the planes then the line of intersection is perpendicular to both normals. I.e. it has direction vector \(\mathbf{N}_1 \times \mathbf{N}_2\)

![Intersection of two planes]

**Example:** Find the intersection of the planes \(x + y + z = 1\) and \(y = 2\).

Normals: \(\mathbf{N}_1 = \langle 1, 1, 1 \rangle\) and \(\mathbf{N}_2 = \langle 0, 1, 0 \rangle\).

Direction vector: \(\mathbf{v} = \mathbf{N}_1 \times \mathbf{N}_2 = \mathbf{i} - \mathbf{k} = \langle 1, 0, -1 \rangle\).

One point of intersection: (by elimination) \(y = 2 \Rightarrow x + 2 + z = 3 \Rightarrow P = (1, 2, 0)\).

Answer: The intersection is the line \(\langle 1, 2, 0 \rangle + t\langle 1, -1, 0 \rangle\).