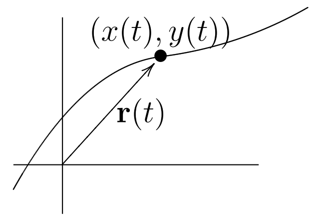
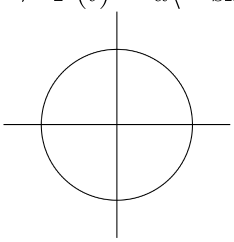
18.02A Topic 22: Vector derivatives: velocity, curvature (2 hours).
Read: TB: 17.4, 17.5



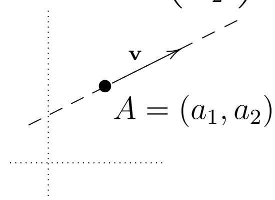
General parametric curve
Think of it as a point moving in time.
$\mathbf{r}(t) = x(t)\,\widehat{\mathbf{i}} + y(t)\,\widehat{\mathbf{j}} = \langle x(t), y(t)\rangle$ = **position vector**

Examples: (from last time)

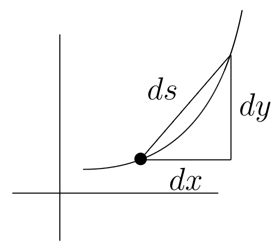
Circle: $\mathbf{r}(t) = a\langle \cos t, \sin t\rangle$
$\Rightarrow\ \mathbf{r}'(t) = a\langle -\sin t, \cos t\rangle$



Line: $\mathbf{r}(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{A} + t\mathbf{v}$
$\Rightarrow\ \mathbf{r}'(t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{v}$



Arclength:
Geometrically this means choose a starting point P on the curve and refer to points on the curve by their distance along the curve from P. (This is like mileage markers along the highway.)
Notation: s = arclength
Geometric approach: (see picture above)



$$ds = \sqrt{(dx)^2 + (dy)^2}\ \Rightarrow\ \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

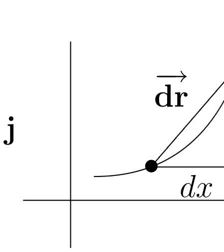
Note: $\dfrac{ds}{dt}$ = speed (has units distance/time).

Example: Find $\dfrac{ds}{dt}$ and arclength for the curve $\mathbf{r}(t) = \langle t, t^2\rangle$ between $(0, 0)$ and $(1, 1)$

$$\frac{ds}{dt} = \sqrt{1 + (2t)^2}\ \Rightarrow\ L = \int_0^1 \sqrt{1 + 4t^2}\,dt.$$

Velocity: If something moves it has a velocity = speed and direction. How do we find it for $\mathbf{r}(t)$? What does it mean geometrically?

Answer: Instantaneous velocity $= \mathbf{r}'(t) = \dfrac{d\mathbf{r}}{dt} = \langle x'(t), y'(t)\rangle$
Reasons:
Over a small time dt the point moves a (vector) $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ and its velocity (displacement/time) is $\dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j}$
Note: speed $= \frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right|$.



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Tangent vector: In the picture above, we see that as Δt shrinks to 0 the vector $\frac{\Delta \mathbf{r}}{\Delta t}$ becomes tangent to the curve
When the parameter is t we can refer to $\mathbf{r}'(t)$ as the 'velocity'. In general, the derivative is given its geometric name: the 'tangent vector'.

Physics approach: (get same formulas)

$$\frac{ds}{dt} = \text{speed} = |\mathbf{r}'(t)| = |\langle x'(t), y'(t)\rangle| = \sqrt{(x')^2 + (y')^2}.$$

Unit tangent vector
The unit vector in the direction of the tangent vector is denoted $\mathbf{T} = \frac{\mathbf{r}(t)}{|\mathbf{r}'(t)|}$.
It's called the unit tangent vector.
Note $\frac{ds}{dt}\mathbf{T} = \mathbf{r}'(t)$.

Nomenclature summary:
Here are a list of names and formulas. We will motivate and derive them below.

$\mathbf{r}(t)$ = position.

s = arclength, speed $= v = \frac{ds}{dt}$.

$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{ds}{dt}\,\mathbf{T}$ = tangent vector, velocity.

$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$ = acceleration.

$\mathbf{T}$ = unit tangent vector, $\mathbf{N}$ = unit normal vector.

κ = curvature, $R = 1/\kappa$ = radius of curvature.

φ = tangent angle.

C = Center of curvature = center of best fitting circle (has radius = radius of curvature).

Formulas: (explained in the following pages)

1. Speed $= \dfrac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{(x')^2 + (y')^2}$.

2. $\mathbf{v} = \dfrac{ds}{dt}\mathbf{T}$, $\mathbf{T} = \dfrac{\mathbf{v}}{ds/dt}$

3. $\mathbf{a}(t) = \dfrac{d^2 s}{dt^2}\mathbf{T} + \kappa\left(\dfrac{ds}{dt}\right)^2\mathbf{N} = \dfrac{d^2 s}{dt^2}\mathbf{T} + \dfrac{v^2}{R}\mathbf{N}$

4. $\kappa = \dfrac{d\mathbf{T}}{ds} = \dfrac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3}$.

4a. For plane curves $\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$: $\kappa = \dfrac{|x''y' - x'y''|}{((x')^2 + (y')^2)^{3/2}}$.

5. $\mathbf{v}\times(\mathbf{a}\times\mathbf{v}) = \kappa v^4\mathbf{N}$.

6. $C = \mathbf{r} + R\,\mathbf{N} = \mathbf{r} + \dfrac{1}{\kappa}\,\mathbf{N}$.

(continued)

Example: Cycloid $\mathbf{r}(\theta) = a\langle \theta - \sin\theta, 1 - \cos\theta \rangle$

$$\frac{d\mathbf{r}}{d\theta} = a\langle 1 - \cos\theta, \sin\theta \rangle = 2a\langle \sin^2\theta/2, \sin\theta/2\cos\theta/2 \rangle$$

$$\left|\frac{d\mathbf{r}}{d\theta}\right| = \frac{ds}{d\theta} = 2a\sqrt{\sin^2\theta/2} = 2a|\sin\theta/2|$$

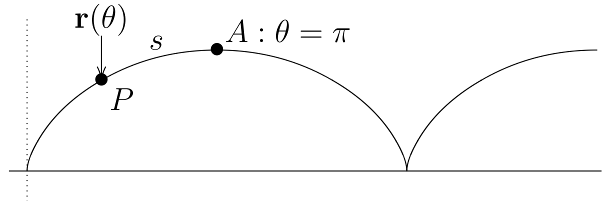
At the cusp $ds/d\theta = 0$, i.e., physically, stop to make a sudden 180 degree turn.

For one arch, $0 < \theta < 2\pi$, $\frac{ds}{d\theta} = 2a\sin\theta/2$

$$\mathbf{T} = \frac{2a\langle \sin^2\theta/2, \sin\theta/2\cos\theta/2 \rangle}{2a\sin\theta/2} = \langle \sin\theta/2, \cos\theta/2 \rangle \text{ (a unit vector!)}$$

$$\begin{aligned} |AP| &= \int_0^\theta \frac{ds}{d\theta}\, d\theta \\ &= -4a\cos\theta/2\big|_0^\theta \\ &= 4a\cos\theta/2 \end{aligned}$$

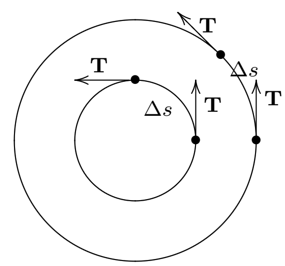
$\Rightarrow$ $|OA| = 4a$ (Wren's theorem)



Curvature: How sharply curved is the trajectory?

That is, how fast does the tangent vector turn in per unit arclenth?
This is tricky so pay attention. Curvature is the rate $\mathbf{T}$ is turning per unit arclength. That is, $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$.

(Smaller circle = faster turning = greater curvature.)



Note well, curvature is a geometric idea– we measure the rate with respect to arclength. The speed the point moves over the trajectory is irrelevant.

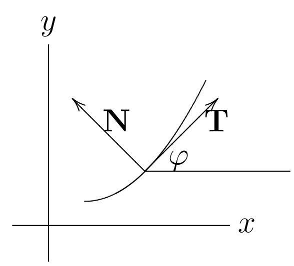
$\mathbf{T}$ is a unit vector $\Rightarrow$ $\mathbf{T} = \langle \cos\varphi, \sin\varphi \rangle$ where φ is the tangent angle.

$$\Rightarrow \frac{d\mathbf{T}}{ds} = \frac{d}{ds}\langle \cos\varphi, \sin\varphi \rangle = \frac{d\varphi}{ds}\langle -\sin\varphi, \cos\varphi \rangle.$$

Both magnitude and direction of $\frac{d\mathbf{T}}{ds}$ are useful:

Curvature $= \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\varphi}{ds}\right|$.

Direction $= \mathbf{N}$ = unit normal $= \langle -\sin\varphi, \cos\varphi \rangle \perp \mathbf{T}$.



Note: the book doesn't use the absolute value in its definition of κ, but it's more standard to include it.

Example: Circle $\mathbf{r}(t) = b(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$

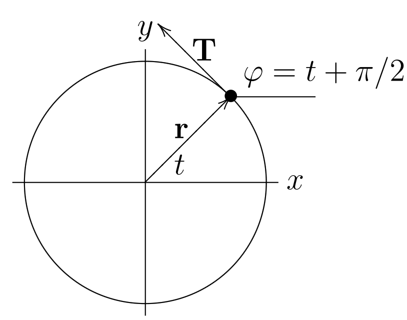
$\mathbf{r}'(t) = b(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})$

$\Rightarrow$ $|\mathbf{v}| = \frac{ds}{dt} = |\mathbf{r}'(t)| = b$ and $\varphi = t + \pi/2$

$\Rightarrow$ $\frac{d\varphi}{ds} = \frac{d\varphi}{dt} \cdot \frac{dt}{ds} = \frac{d\varphi/dt}{ds/dt} = \frac{1}{b}$

I.e. curvature of a circle = 1/radius (bigger circle = smaller curvature).

Using formula 4: $\mathbf{a} = -b(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$ $\Rightarrow$ $\kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = b^2/b^3 = 1/b$.



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Proofs of formulas 3-5:
Note: we will repeatedly use that $v = \frac{ds}{dt}$.
Formula 3 is an application of the product and chain rules:
Start with $\mathbf{v} = \dfrac{ds}{dt}\mathbf{T}$.

$$\begin{aligned} \Rightarrow \ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt} \\ &= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N}. \end{aligned}$$

In physics this is the decomposition of acceleration into tangential and radial components.

Formula 4 now follows from formula 3 since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors:

$$\mathbf{a} \times \mathbf{v} = \left(\frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N}\right) \times \frac{ds}{dt}\mathbf{T} = \left(\frac{ds}{dt}\right)^3 \kappa(\mathbf{N} \times \mathbf{T}).$$

Since $\mathbf{N}$ and $\mathbf{T}$ are orthogonal unit vectors $\mathbf{N} \times \mathbf{T}$ is a unit vector
$\Rightarrow \ |\mathbf{a} \times \mathbf{v}| = (\frac{ds}{dt})^3\kappa = v^3\kappa.$ ■

The second part of formula 4 is just the first in coordinates:
$\mathbf{v} = x'\,\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}}$ and $\mathbf{a} = x''\,\hat{\mathbf{i}} + y''\,\hat{\mathbf{j}}$
$\Rightarrow \ \mathbf{a} \times \mathbf{v} = (x''y' - x'y'')\hat{\mathbf{k}}$ and $v = \sqrt{(x')^2 + (y')^2}$
$\Rightarrow$ what we want.

Formula 5 now follows from what we just did. We found
$\mathbf{a} \times \mathbf{v} = v^3\kappa(\mathbf{N} \times \mathbf{T}). \ \Rightarrow \ \mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = v^4\kappa\,\mathbf{T} \times (\mathbf{N} \times \mathbf{T}) = v^4\kappa\,\mathbf{N}.$
The last equality is easy using your right hand (since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors).

Example: Find the curvature of the cycloid $\mathbf{r}(\theta) = a(\theta - \sin\theta)\,\hat{\mathbf{i}} + a(1 - \cos\theta)\,\hat{\mathbf{j}}$
$\mathbf{v} = a(1 - \cos\theta)\,\hat{\mathbf{i}} + a\sin\theta\,\hat{\mathbf{j}}$ and $\mathbf{a} = a\sin\theta\,\hat{\mathbf{i}} + a\cos\theta\,\hat{\mathbf{j}}$.
$\Rightarrow \ \mathbf{a}\times\mathbf{v} = a^2((1-\cos\theta)\cos\theta - \sin\theta\sin\theta)\,\hat{\mathbf{k}} = a^2(\cos\theta - 1)\,\hat{\mathbf{k}}$ and $v = a\sqrt{2(1 - \cos\theta)}$
$\Rightarrow \ \kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{a^2(1-\cos\theta)}{a^3 2^{3/2}(1-\cos\theta)^{3/2}} = \frac{1}{a2^{3/2}\sqrt{(1-\cos\theta)}} = \frac{1}{4a|\sin(\theta/2)|}$.

Example: Circle with different parameterization. $\mathbf{r}(t) = b(\cos t^2, \sin t^2)$
Since curvature is geometric it should be independent of parameterization. We'll use formula 4 to verify this in this example.
$\mathbf{v} = \langle -2bt\sin t^2, 2bt\cos t^2\rangle$, $\mathbf{a} = \langle -2b\sin t^2 - 4bt^2\cos t^2, 2b\cos t^2 - 4bt^2\sin t^2\rangle$.
A little algebra $\Rightarrow$ $\mathbf{a} \times \mathbf{v} = -8b^2t^3\,\mathbf{k}$ and $|\mathbf{v}| = 2bt$ $\Rightarrow$ $\kappa = \frac{8b^2t^3}{8b^3t^3} = \frac{1}{b}$. (same as before!).

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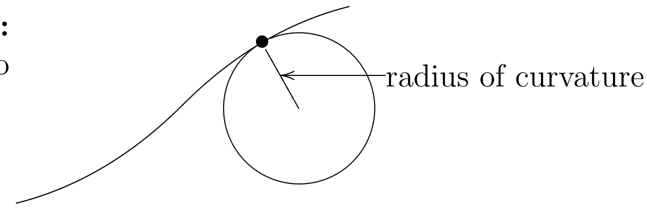
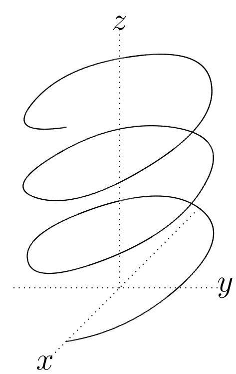
These circle examples ($\kappa = 1/\text{radius}$) explains the following definition
Radius of curvature $= \frac{1}{\kappa}$

The center of curvature and the osculating circle:
The osculating (kissing) circle is the best fitting circle to the curve.
Radius = radius of curvature.
Center along normal direction.



Example: For the helix $\mathbf{r}(t) = \cos t\,\widehat{\mathbf{i}} + \sin t\,\widehat{\mathbf{j}} + at\,\widehat{\mathbf{k}}$ find the radius of curvature and center of curvature for arbitrary t.



<u>**answer:**</u> We will use the formulas (2), (3) and (4),

$\mathbf{v} = -\sin t\,\widehat{\mathbf{i}} + \cos t\,\widehat{\mathbf{j}} + a\,\widehat{\mathbf{k}}$; $\quad \mathbf{a} = -\cos t\,\widehat{\mathbf{i}} - \sin t\,\widehat{\mathbf{j}}$.

$\Rightarrow\ |\mathbf{v}| = \sqrt{1+a^2}$; $\quad \mathbf{a}\times\mathbf{v} = -a\sin t\,\widehat{\mathbf{i}} + a\cos t\,\widehat{\mathbf{j}} - \widehat{\mathbf{k}}$.

Formula (4) $\Rightarrow\ \kappa = \frac{|\mathbf{a}\times\mathbf{v}|}{|\mathbf{v}|^3} = \frac{\sqrt{1+a^2}}{(1+a^2)^{3/2}} = \frac{1}{1+a^2}$.

$\Rightarrow$ <u>radius of convergence</u> $= R = 1 + a^2$.

The center of curvature $C = \mathbf{r}(t) + R\mathbf{N}\ \Rightarrow$ we have to find $R\mathbf{N}$.

Since we already have $\mathbf{a}\times\mathbf{v}$ we *could* use formula (5).

Instead we note that $\frac{ds}{dt} = \sqrt{1+a^2}\ \Rightarrow\ \frac{d^2s}{dt^2} = 0\ \Rightarrow\ \mathbf{a} = \kappa(\frac{ds}{dt})^2\mathbf{N}$.

But $\mathbf{a}$ is already a unit vector $\Rightarrow\ \mathbf{a} = \mathbf{N}$.

$\Rightarrow\ R\mathbf{N} = -(1+a^2)(\cos t\,\widehat{\mathbf{i}} + \sin t\,\widehat{\mathbf{j}})$.

$$\begin{aligned}\Rightarrow\ C &= (\cos t, \sin t, t) - ((1+a^2)\cos t, (1+a^2)\sin t, 0)\\ &= (-a^2\cos t, -a^2\sin t, t).\end{aligned}$$

Example: For the parabola $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ find $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$, ds/dt, κ, R, $\mathbf{N}$ and C for arbitrary t.

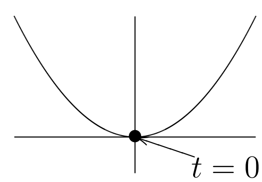
$\mathbf{v} = \mathbf{i} + 2t\,\mathbf{j}$, $\quad \mathbf{a} = 2\mathbf{j}$.

$\Rightarrow\ ds/dt = |\mathbf{v}| = \sqrt{1+4t^2}$, $\quad \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+4t^2}}\,\mathbf{i} + \frac{2t}{\sqrt{1+4t^2}}\,\mathbf{j}$.

Formula 4: $\quad \mathbf{a}\times\mathbf{v} = -2\mathbf{k}. \quad \Rightarrow\ \kappa = \frac{2}{(1+4t^2)^{3/2}}$.

(Maximum curvature at $t = 0$ as expected.)

$R = 1/\kappa = \frac{(1+4t^2)^{3/2}}{2}$.



Formula 5: $\quad \mathbf{v}\times(\mathbf{a}\times\mathbf{v}) = -4t\,\mathbf{i} + 2\,\mathbf{j}.\ \Rightarrow\ \mathbf{N} = \frac{1}{2\sqrt{1+4t^2}}(-4t\,\mathbf{i} + 2\,\mathbf{j})$.

Formula 6: $\quad C = \mathbf{r} + R\mathbf{N} = (t\,\mathbf{i} + t^2\,\mathbf{j}) + \frac{1+4t^2}{4}(-4t\,\mathbf{i} + 2\,\mathbf{j})$.