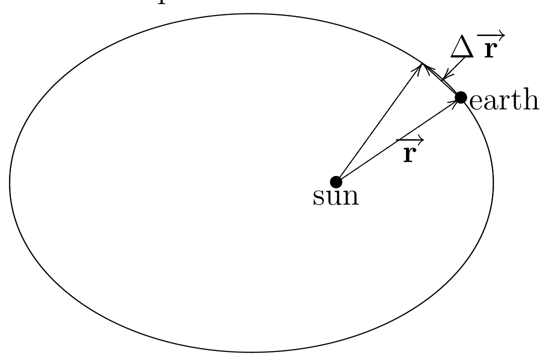
18.02A Topic 23: Continuation, Kepler's second law.
Read: SN: K

What I have here is just a bad version of the notes §K. I suggest you read that instead.

Claim: If a body moves under a central force then it sweeps out equal areas in equal time.

Proof:
Note a central force means $\overrightarrow{\mathbf{r}}$ is parallel to $\overrightarrow{\mathbf{a}}$.



In a short time Δt the position vector sweeps out an area ΔA.
Using vectors we see $\Delta A \approx \frac{1}{2}|\overrightarrow{\mathbf{r}} \times \Delta\overrightarrow{\mathbf{r}}|$.
$\Rightarrow \frac{dA}{dt} = \frac{1}{2}|\overrightarrow{\mathbf{r}} \times \frac{d\overrightarrow{\mathbf{r}}}{dt}|$.
Equal areas in equal time $\Leftrightarrow \frac{dA}{dt}$ = constant.
Consider $\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{r}} \times \frac{d\overrightarrow{\mathbf{r}}}{dt}$.

$$\text{The product rule } \Rightarrow \frac{d\overrightarrow{\mathbf{w}}}{dt} = \frac{d\overrightarrow{\mathbf{r}}}{dt} \times \frac{d\overrightarrow{\mathbf{r}}}{dt} + \overrightarrow{\mathbf{r}} \times \frac{d^2\overrightarrow{\mathbf{r}}}{dt^2} = \frac{d\overrightarrow{\mathbf{r}}}{dt} \times \frac{d\overrightarrow{\mathbf{r}}}{dt} + \overrightarrow{\mathbf{r}} \times \overrightarrow{\mathbf{a}}$$

Both terms are 0 since $\overrightarrow{\mathbf{a}}$ is parallel to $\overrightarrow{\mathbf{r}}$.
$\frac{d\overrightarrow{\mathbf{w}}}{dt} = 0 \Rightarrow \frac{dA}{dt}$ = constant. ■