18.02A Topic 24: Functions of several variables, partial derivatives.
Read: TB: 19.1

18.02A Topic 25: Tangent plane, level curves, contour surfaces.
Read: TB: 19.2 SN: TA

Examples: Functions of several variables
$f(x,y) = x^2 + y^2 \;\Rightarrow\; f(1,2) = 5$ etc.
$f(x,y) = xy^2 e^{x+y}$
$f(x,y,z) = xy \log z$
Ideal gas law: $P = kT/V$

Dependent and independent variables
In $z = f(x,y)$ we say x, y are independent and z is dependent.
This indicates that x and y are free to take values and then z depends on these values.
For now it will be clear which are which, later we'll have to think more care.

Partial derivatives
$\dfrac{\partial f}{\partial x}$ differentiate with respect to x holding all other independent variables fixed –i.e. pretend they are constant.

Example: $f(x,y) = x^2y + y^2 + x^2 - 3 \;\Rightarrow\; \frac{\partial f}{\partial x} = 2xy + 2x, \quad \frac{\partial f}{\partial y} = x^2 + 2y.$

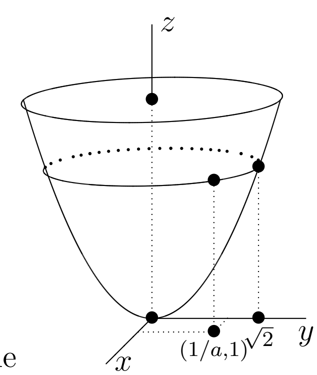
Notation: $z = f(x,y)$
$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x. \qquad \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = f_x(x_0,y_0) = \frac{\partial f}{\partial x}\Big|_0 = z_x(x_0,y_0).$

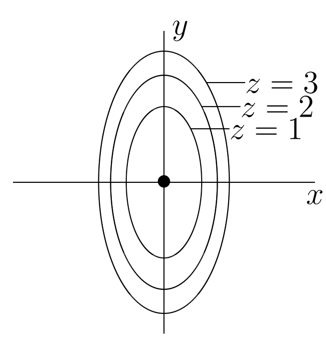
Example: $f(x,y) = x^2y + y^2 + x^2 - 3 \;\Rightarrow\; \frac{\partial f}{\partial x} = 2xy + 2x, \quad \frac{\partial f}{\partial x}\Big|_{(1,2)} = 2\cdot1\cdot2 + 2\cdot1 = 6.$

Graphs and level curves
$y = f(x)$: 1 independent variable, 1 dependent variable $\Rightarrow$ need 2 dimensions to graph
Graphing technique:
go to x then compute $y = f(x)$ then go up to height y.

$z = f(x,y)$ is the same.
Example: $z = f(x,y) = a^2x^2 + y^2$
Graph:
Go to (x,y)
compute $z = f(x,y)$
go up to height z

Level curve at height 3:
Slice graph (surface) at height $z = 3$
look down to $x - y$ plane
I.e. level curve is $z = 3 = a^2x^2 + y^2$
i.e. level curve = an ellipse in $x-y$ plane

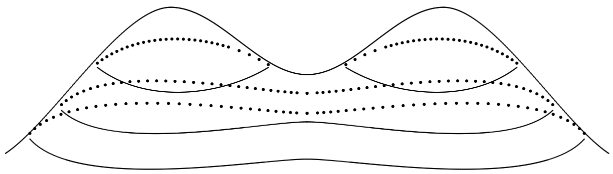




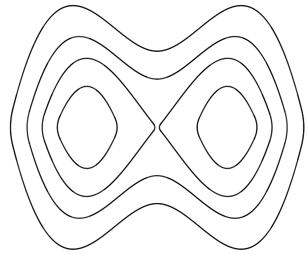
Plot of level curves 'pulls out' to the 3 dimensional graph.

(continued)

Idea is same as for topographic maps



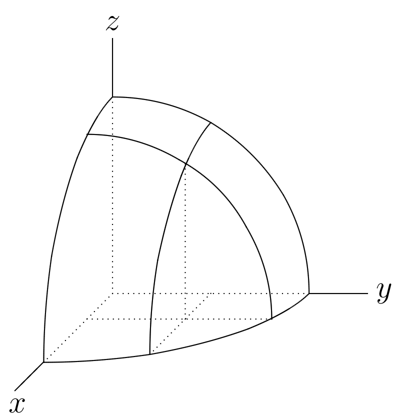
Mountain pass

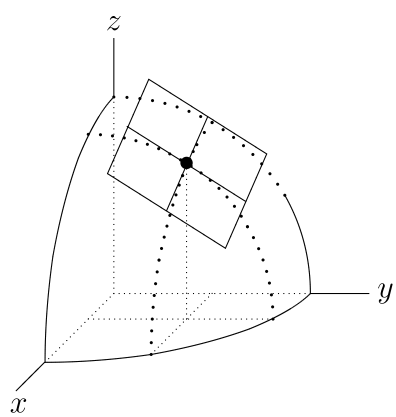


Level curves

Contours:
Contours are the curves *on the graph* at a given height. They sit above the level curves.
Remember: level curves are in the plane, contours are on the graph.

Geometry:
Tangent plane -tangent to surface, must contain 2 tangent lines.





Theorem: (Assuming it's not vertical) the equation of the tangent plane is $z - z_0 = \left.\frac{\partial z}{\partial x}\right|_0 (x - x_0) + \left.\frac{\partial z}{\partial y}\right|_0 (y - y_0)$

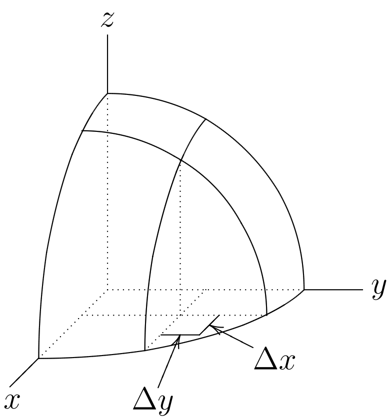
proof: Two tangent lines $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ z_x(x_0, y_0) \end{pmatrix}$ and $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ z_y(x_0, y_0) \end{pmatrix}$

$\Rightarrow$ Normal $\mathbf{N} = (1, 0, \left.\frac{\partial z}{\partial x}\right|_0) \times (1, 0, \left.\frac{\partial z}{\partial y}\right|_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & z_x \\ 1 & 0 & z_y \end{vmatrix} = (-z_x, -z_y, 1)$

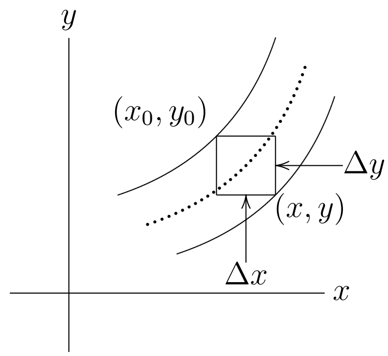
$\Rightarrow$ plane $= -z_x \cdot (x - x_0) - z_y \cdot (y - y_0) + (z - z_0) = 0$

(continued)

Approximation formula: Tangent plane approximates surface
$\Delta x = (x - x_0),\ \Delta y = (y - y_0),\ \Delta z = f(x, y) - f(x_0, y_0)$
$\Delta z \approx \left.\frac{\partial z}{\partial x}\right|_0 \Delta x + \left.\frac{\partial z}{\partial y}\right|_0 \Delta y$



Graph of $z = f(x, y)$.



Level curves.

Example: Suppose you have a box of dimensions 5, 10 and 15 cm. Use the tangent plane approximation formula to estimate the percentage change in volume if each of the dimensions is increased by 0.5 cm.
answer: Label the sides of the box, x, y, z.
$\Rightarrow$ Volume $= V = xyz \ \Rightarrow\ \frac{\partial V}{\partial x} = yz; \quad \frac{\partial V}{\partial y} = xz; \quad \frac{\partial V}{\partial z} = zy.$
$\Rightarrow\ \left.\frac{\partial V}{\partial x}\right|_{(5,10,15)} = 150, \quad \left.\frac{\partial V}{\partial y}\right|_{(5,10,15)} = 75, \quad \left.\frac{\partial V}{\partial z}\right|_{(5,10,15)} = 50.$
$\Rightarrow\ \Delta V \approx 150\Delta x + 75\Delta y + 50\Delta z.$
Thus, $\Delta x = \Delta y = \Delta z = 0.5 \ \Rightarrow\ \Delta V \approx (150 + 75 + 50) \cdot 0.5 = 137.5.$
$\Rightarrow$ Percentage change $= \Delta V/V = 137.5/750 = 18.3\%$.

In the example, volume is most sensitive to changes in which side?
answer: The side of length 5 since Δx has the biggest coefficient in the approximation formula.