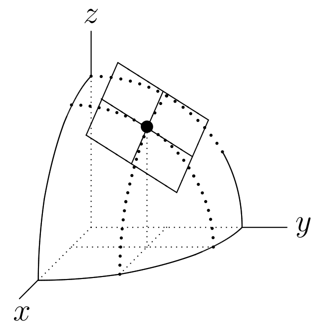
**18.02A Topic 26**: Tangent plane approximation, directional derivatives.
Read: TB: 19.1, 19.2 SN: TA




From last time we have the tangent plane approximation.
$\Delta x = (x - x_0),\ \Delta y = (y - y_0),\ \Delta w = f(x, y) - f(x_0, y_0)$
$\Delta w \approx \frac{\partial w}{\partial x}\Big|_0 \Delta x + \frac{\partial w}{\partial y}\Big|_0 \Delta y$
The supplemental notes §TA give an analytic argument for this and its genaralization to 3 dimensions:
Suppose $w = f(x, y, z)$ and
$\Delta x = (x - x_0),\ \Delta y = (y - y_0),\ \Delta z = (z - z_0)$ then
$\Delta w = f(x, y) - f(x_0, y_0) \approx \frac{\partial w}{\partial x}\Big|_0 \Delta x + \frac{\partial w}{\partial y}\Big|_0 \Delta y + \frac{\partial w}{\partial z}\Big|_0 \Delta z$

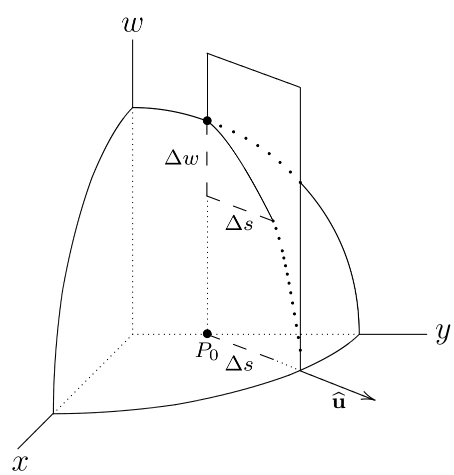
**Gradient**

$\frac{\partial w}{\partial x}\Big|_0$ = rate of change in $\hat{\mathbf{i}}$ direction, $\frac{\partial w}{\partial y}\Big|_0$ = rate of change in $\hat{\mathbf{j}}$ direction.

$$\boxed{\left\langle \frac{\partial w}{\partial x}\Big|_0, \frac{\partial w}{\partial y}\Big|_0 \right\rangle = \textbf{gradient} \text{ of } w = \boldsymbol{\nabla} w}$$

To evaluate at $P_0$ we write $\boldsymbol{\nabla} w(P_0)$. (Will use this in a moment.)

**Directional derivative**



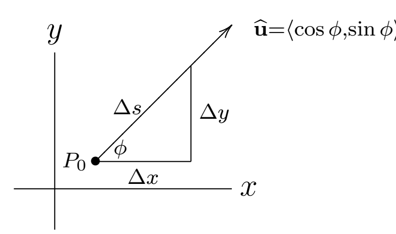

Fix a direction $\hat{\mathbf{u}}$ and a point $P_0$ in the *plane.*

The **directional derivative** of $w$ at $P_0$ in the direction $\hat{\mathbf{u}}$ is defined as

$$\boxed{\frac{dw}{ds}\Big|_{P_0, \hat{\mathbf{u}}} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s}.}$$

Here $\Delta w$ is the change in $w$ caused by a step of length $\Delta s$ in the direction of $\hat{\mathbf{u}}$ (all in the $xy$-plane).

It is a fact that: $\boxed{\frac{dw}{ds}\Big|_{P_0, \hat{\mathbf{u}}} = \boldsymbol{\nabla} w(P_0) \cdot \hat{\mathbf{u}}.}$




**Proof:** The tangent plane approximation and the bottom picture at right show
$\frac{\Delta w}{\Delta s} \approx \frac{\partial w}{\partial x}\Big|_0 \frac{\Delta x}{\Delta s} + \frac{\partial w}{\partial y}\Big|_0 \frac{\Delta y}{\Delta s} \approx \frac{\partial w}{\partial x}\Big|_0 \cos\phi + \frac{\partial w}{\partial y}\Big|_0 \sin\phi$
But $\hat{\mathbf{u}} = \langle \cos\phi, \sin\phi \rangle$ since it is a unit vector. Thus, the last formula is just $\boldsymbol{\nabla} w \cdot \hat{\mathbf{u}}$. In the limit the approximations become exact and we get the boxed equation. QED

*(continued)*

**Example:** (Algebraic example) Let $w = x^3 + 3y^2$.

Compute $\dfrac{dw}{ds}$ at $P_0 = (1, 2)$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$.

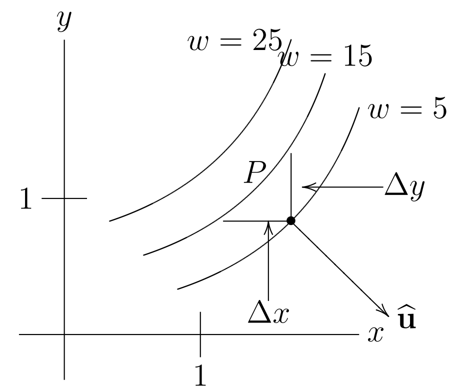
i) $\boldsymbol{\nabla} w = \langle 3x^2 + 3y^2\rangle \;\Rightarrow\; \boldsymbol{\nabla} w|_{(1,2)} = \langle 15, 12\rangle = 15\,\mathbf{i} + 12\,\mathbf{j}$.

ii) $\widehat{\mathbf{u}} = \dfrac{\mathbf{v}}{|\mathbf{v}} = \dfrac{3}{5}\,\mathbf{i} + \dfrac{4}{5}\,\mathbf{j}$.

iii) $\left.\dfrac{dw}{ds}\right|_{P_0,\widehat{\mathbf{u}}} = \boldsymbol{\nabla} w|_{(1,2)} \cdot \widehat{\mathbf{u}} = (15\,\mathbf{i} + 12\,\mathbf{j}) \cdot (\dfrac{3}{4}\,\mathbf{i} + \dfrac{4}{5}\,\mathbf{j}) = \boxed{\dfrac{93}{5}.}$

**Example:** (Geometric example) Let $\widehat{\mathbf{u}}$ be the direction of $\langle 1, -1\rangle$.

Using the picture at right estimate $\left.\dfrac{\partial w}{\partial x}\right|_P$, $\left.\dfrac{\partial w}{\partial y}\right|_p$, and $\left.\dfrac{dw}{ds}\right|_{P,\widehat{\mathbf{u}}}$.




By measuring from $P$ to the next in level curve in the $x$ direction we see that $\Delta x \approx -.5$.

$\Rightarrow \boxed{\left.\dfrac{\partial w}{\partial x}\right|_P \approx \dfrac{\Delta w}{\Delta x} \approx \dfrac{10}{-.5} = -20.}$

Similarly, we get $\boxed{\left.\dfrac{\partial w}{\partial y}\right|_P \approx 20..}$

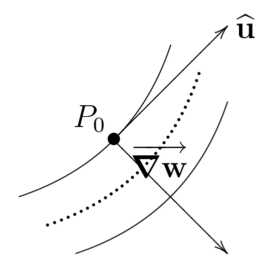
Measuring in the $\mathbf{u}$ direction we get $\Delta s \approx -.3$

$\Rightarrow \boxed{\left.\dfrac{dw}{ds}\right|_{P,\widehat{\mathbf{u}}} \approx \dfrac{\Delta w}{\Delta s} \approx \dfrac{10}{.3} = -33.3.}$

**Direction of maximum change**: $\widehat{\mathbf{u}} = \dfrac{\boldsymbol{\nabla} w}{|\boldsymbol{\nabla} w|}$ (Proof: angle = 0).

**The Gradient is perpendicular to level curves**

$\boxed{\boldsymbol{\nabla} w \perp \text{ level curve (surface) } w = c.}$




**Example:** Consider the graph of $y = \mathrm{e}^x$. Find a vector perpindicular to the tangent to at the point $(1, \mathrm{e})$.

Old method: Find the slope take the negative reciprocal and make the vector.
New method: This graph is the level curve of $w = y - \mathrm{e}^x$ with $w = 0$.
$\boldsymbol{\nabla} w = \langle -\mathrm{e}^x, 1\rangle \;\Rightarrow\;$ normal $= \boldsymbol{\nabla} w(1, e) = \langle -\mathrm{e}, 1\rangle$.

**proof:** If $\widehat{\mathbf{u}}$ is tangent to the level curve at $P_0$ then $\left.\dfrac{dw}{ds}\right|_{P_0,\widehat{\mathbf{u}}} = 0$

since $w$ is constant along the level curve, i.e., $\left.\dfrac{dw}{ds}\right|_{P_0,\widehat{\mathbf{u}}} = \boldsymbol{\nabla} w(P_0) \cdot \widehat{\mathbf{u}} = 0.$ QED

*(continued)*

**Example:** Find the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $P = (1, 1, 1)$.

Introduce a new variable $w = x^2+2y^2+3z^2 = 6$. Our surface is the level surface $w = 6$ $\Rightarrow$ normal to surface is $\nabla w = \langle 2x, 4y, 6z \rangle$. At the point $P$ we have $\nabla w|_P = \langle 2, 4, 6 \rangle$. Using point normal form the equation of the tangent plane is $2(x - 1) + 4(y - 1) + 6(z - 1) = 0 \;\Leftrightarrow\; \boxed{2x + 4y + 6z = 12.}$