18.02A Topic 27: Chain rule.
Read: TB: 19.6

Tangent plane approximation formula:
$w = f(x, y)$: $\Rightarrow$ $\Delta w \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$.

Single variable approximation and chain rule:

Approximation formula for $y = f(x)$: $\quad \Delta y \approx \dfrac{dy}{dx}\Delta x$.

If x is a function of t then divide the approximation formula by Δt $\Rightarrow$ $\dfrac{\Delta y}{\Delta t} = \dfrac{df}{dx}\dfrac{\Delta x}{\Delta t}$.

In the limit this becomes the chain rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$.

Example: $f(x) = x^3,\ x(t) = \sin t \ \Rightarrow\ f(x(t)) = \sin^3 t \ \Rightarrow\ \dfrac{df}{dt} = 3x^2 \cos t = 3\sin^2 t \cos t$.

Multivariable functions:
Suppose $w = f(x, y)$ and $x = x(u, v),\ y = y(u, v)$.
Dependent variable = w, independent variables = u, v, intermediate variables = x, y.

Multivariable chain rule

Likewise we get the multivariable chain rule by, for example, holding v constant and dividing the tangent plane approximation formula by Δu.

Approximation formula: $\Delta w = \dfrac{\partial w}{\partial x}\Delta x + \dfrac{\partial w}{\partial y}\Delta y \ \Rightarrow\ \dfrac{\Delta w}{\Delta u} = \dfrac{\partial w}{\partial x}\dfrac{\Delta x}{\Delta u} + \dfrac{\partial w}{\partial y}\dfrac{\Delta y}{\Delta u}$

Letting $\Delta u \to 0$ gives the chain rule for $\frac{\partial w}{\partial u}$:

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}}$$

Example: Given $w = x^2 y + y^2 + x, \quad x = u^2 v, \quad y = uv^2$ find $\dfrac{\partial w}{\partial u}$.

($\Rightarrow$ u, v independent, x, y intermediate, w dependent.)

$\dfrac{\partial w}{\partial x} = 2xy + 1, \quad \dfrac{\partial w}{\partial y} = x^2 + 2y, \quad \dfrac{\partial x}{\partial u} = 2uv, \quad \dfrac{\partial y}{\partial u} = v^2, \quad \dfrac{\partial x}{\partial v} = u^2, \quad \dfrac{\partial y}{\partial v} = 2uv.$

$$\begin{aligned}
\Rightarrow \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \\
&= (2xy + 1)2uv + (x^2 + 2y)v^2 (2u^2 v \cdot uv^2 + 1)2uv + (u^4 v^2 + 2uv^2)v^2 \\
&= 5u^4 v^4 + 2uv + 2uv^4.
\end{aligned}$$

Check: $w = x^2 y + y^2 + x = u^5 v^4 + u^2 v^4 + u^2 v \ \Rightarrow\ \dfrac{\partial w}{\partial u} = 5u^4 v^4 + 2uv^4 + 2uv.$ ■

(continued)

Special case example:

$w = F(x, y, z) = x^2 + y^3 + z^4, \quad (x(t), y(t), z(t)) = (\cos t, t, \sin t).$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = 2x \cdot (-\sin t) + 3y^2 \cdot 1 + 4z^3 \cdot \cos t.$

(We leave it in this implicit ('mixed') form. It could be written out all in terms of t.)

Associated story: The temperature in space varies and is given by the function $T(x, y, z)$. An ant crawls along a wire whose shape is described by $\mathbf{r}(t) = (x(t), y(t), z(t))$. What is the rate of change of temperature experienced by the ant. (You might not care but the ant certainly does.)

Answer: $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = \nabla T \cdot \frac{d\mathbf{r}}{dt}.$

Theoretical example:
Suppose $w = f(x, y, z)$ and P_0 is on the level surface $w(x, y, z) = c$.
Show $\nabla w(P_0)$ is perpendicular to the level surface.
Answer: Draw any curve on the surface $\mathbf{r}(t) = (x(t), y(t), z(t))$ such the $\mathbf{r}(0) = P_0$.

$\Rightarrow \; w(t) = f(x(t), y(t), y(t)) = c$

$\Rightarrow \; \frac{dw}{dt} = 0 = \nabla w(P_0) \cdot \mathbf{r}'(0)$

$\Rightarrow \; \nabla w(P_0)$ is perpendicular to any curve on the surface through P_0. QED

Ambiguous notation

Often you have to figure out the dependent and independent variables from context. Thermodynamics is a big culprit here:
Variables: P, T, V, U, S. *Any* two can be taken to be independent and the others are functions of those two.

We will do more with this in the future.