**18.02A Topic 27: Chain rule.**
Read: TB: 19.6

**Tangent plane approximation formula:**
\[ w = f(x, y) \Rightarrow \Delta w \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y. \]

**Single variable approximation and chain rule:**
Approximation formula for \( y = f(x) \):
\[ \Delta y \approx \frac{dy}{dx} \Delta x. \]

If \( x \) is a function of \( t \) then divide the approximation formula by \( \Delta t \):
\[ \frac{\Delta y}{\Delta t} = \frac{df}{dx} \frac{\Delta x}{\Delta t}. \]

In the limit this becomes the chain rule:
\[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}. \]

**Example:** \( f(x) = x^3, \ x(t) = \sin t \Rightarrow f(x(t)) = \sin^3 t \Rightarrow \frac{df}{dt} = 3x^2 \cos t = 3\sin^2 t \cos t. \)

**Multivariable functions:**
Suppose \( w = f(x, y) \) and \( x = x(u, v), \ y = y(u, v). \)
Dependent variable = \( w \), independent variables = \( u, v \), intermediate variables = \( x, y \).

**Multivariable chain rule:**
Likewise we get the multivariable chain rule by, for example, holding \( \nu \) constant and dividing the tangent plane approximation formula by \( \Delta u \).

Approximation formula:
\[ \Delta w = \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y \Rightarrow \frac{\Delta w}{\Delta u} = \frac{\partial w}{\partial x} \frac{\Delta x}{\Delta u} + \frac{\partial w}{\partial y} \frac{\Delta y}{\Delta u}. \]

Letting \( \Delta u \to 0 \) gives the chain rule for \( \frac{\partial w}{\partial u} \):
\[
\begin{align*}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}, \\
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}.
\end{align*}
\]

**Example:** Given \( w = x^2y + y^2 + x, \ x = u^2v, \ y = uv^2 \) find \( \frac{\partial w}{\partial u} \).

\( \Rightarrow u, v \) independent, \( x, y \) intermediate, \( w \) dependent.
\[
\frac{\partial w}{\partial x} = 2xy + 1, \quad \frac{\partial w}{\partial y} = x^2 + 2y, \quad \frac{\partial x}{\partial u} = 2uv, \quad \frac{\partial y}{\partial u} = v^2, \quad \frac{\partial x}{\partial v} = u^2, \quad \frac{\partial y}{\partial v} = 2uv.
\]
\[
\Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = (2xy + 1)2uv + (x^2 + 2y)v^2(2uv + (u^4v^2 + uv^2))v^2 = 5u^4v^4 + 2uv + 2uv^4.
\]

Check: \( w = x^2y + y^2 + x = u^5v^4 + u^2v^4 + u^2v \Rightarrow \frac{\partial w}{\partial u} = 5u^4v^4 + 2uv^4 + 2uv.\)

(continues)
Special case example:

$$w = F(x, y, z) = x^2 + y^3 + z^4, \quad (x(t), y(t), z(t)) = (\cos t, t, \sin t).$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = 2x \cdot (-\sin t) + 3y^2 \cdot 1 + 4z^3 \cdot \cos t.$$  

(We leave it in this implicit ('mixed') form. It could be written out all in terms of $t$.)

**Associated story:** The temperature in space varies and is given by the function $T(x, y, z)$. An ant crawls along a wire whose shape is described by $r(t) = (x(t), y(t), z(t))$. What is the rate of change of temperature experienced by the ant. (You might not care but the ant certainly does.)

**Answer:**

$$\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{\partial T}{\partial z} \frac{dz}{dt} = \nabla T \cdot \frac{dr}{dt}.$$ 

Theoretical example:

Suppose $w = f(x, y, z)$ and $P_0$ is on the level surface $w(x, y, z) = c$. Show $\nabla w(P_0)$ is perpendicular to the level surface.

**Answer:** Draw any curve on the surface $r(t) = (x(t), y(t), z(t))$ such the $r(0) = P_0$.  

$$\Rightarrow w(t) = f(x(t), y(t), z(t)) = c$$  

$$\Rightarrow \frac{dw}{dt} = 0 = \nabla w(P_0) \cdot r'(0)$$  

$$\Rightarrow \nabla w(P_0)$$ is perpendicular to any curve on the surface through $P_0$. QED

Ambiguous notation

Often you have to figure out the dependent and independent variables from context. Thermodynamics is a big culprit here:

Variables: $P, T, V, U, S$. *Any* two can be taken to be independent and the others are functions of those two.

We will do more with this in the future.