18.02A Topic 28: Max-min problems, least squares.
Read: TB: 19.7, SN: LS

Standard calculus question:
Given $z = f(x, y)$ where are the (relative) maxima and minima?
Answer: At the points where $\nabla f = \langle 0,\, 0\rangle$.
I.e. $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Critical points: Such points are called critical points.

Example: $z = x^2 + y^2$
$\nabla z = \langle 2x,\, 2y\rangle \Rightarrow$ only critical point is $(0, 0)$.
Clearly a minimum.

Example: $z = -x^2 - y^2$
$\nabla z = \langle -2x,\, -2y\rangle \Rightarrow$ only critical point is $(0, 0)$.
Clearly a maximum.

Example: $z = y^2 - x^2$
$\nabla z = \langle -2x,\, 2y\rangle \Rightarrow$ only critical point is $(0, 0)$.
Clearly neither maximum or minimum.

Critical point = horizontal tangent plane
Equation of tangent plane at (x_0, y_0, z_0) $(z_0 = f(x_0, y_0))$ is

$$(z - z_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0).$$

At a critical point this becomes: $z - z_0 = 0$ = horizontal plane.

Example: Making a box.
Sides double thick, bottom triple thick, no top, volume = 3.
What dimensions use the least amount of cardboard?
Area of one side = yz, two sides, double thick $\Rightarrow$ cardboard used = $4yz$.
Area front (and back) = xz, single thick $\Rightarrow$ cardboard used = $2xz$.
Area bottom = xy, triple thick $\Rightarrow$ cardboard used = $3xy$.
Total cardboard used = $w = 4yz + 2xz + 3xy$.
Volume = $3 = xyz \Rightarrow z = \frac{3}{xy}$
$\Rightarrow w = \frac{12}{x} + \frac{6}{y} + 3xy$
Critical points: $w_x = -\frac{12}{x^2} + 3y = 0$, $w_y = -\frac{6}{y^2} + 3x = 0$
$w_x = 0 \Rightarrow y = \frac{4}{x^2}$
$w_y = 0 \Rightarrow -\frac{6}{16}x^4 + 3x = 0$
$\Rightarrow x = 2, y = 1, z = 3/2$.
Clearly a minimum. (physically we know it must have a minimum somewhere, it can't be on the boundary (axes) because w is infinite there $\Rightarrow$ only critical point must be a minimum.

(continued)

Least squares You should read §LS in the notes.

Start with data points: (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n).
Try to fit a line, $y = ax + b$, to the data.
The 'squared error sum' is $E(a,b) = \sum_i (y_i - (ax_i + b))^2$.

The least squares fit is the line that minimizes E.
To find the minimum we set $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$.

$$\frac{\partial E}{\partial a} = \sum -2x_i(y_i - (ax_i + b)) = 2\sum_i ax_i^2 + bx_i - x_i y_i = 0.$$

$$\frac{\partial E}{\partial b} = \sum -2x_i(y_i - (ax_i + b) = 2\sum_i ax_i + b - y_i = 0.$$

This gives the least squares equations for a line:

$$\Rightarrow \quad \begin{aligned} a\sum_i x_i^2 + b\sum_i x_i &= \sum_i x_i y_i \\ a\sum_i x_i + b\,n &= \sum y_i \end{aligned}$$

Example: Use least squares to fit a line to the following data: (0,1), (2,1), (3,4).
<u>answer:</u> In our case, $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, 1)$ and $(x_3, y_3) = (3, 4)$.
$\Rightarrow \sum x_i^2 = 13, \quad \sum x_i = 5, \quad n = 3, \quad \sum x_i y_i = 14, \quad \sum y_i = 6.$
$\Rightarrow 13a + 5b = 14; \quad 5a + 3b = 6 \ \Rightarrow\ a = 6/7$ and $b = 4/7$.
The least squares line has equation $y = \frac{6}{7}x + \frac{4}{7}$.

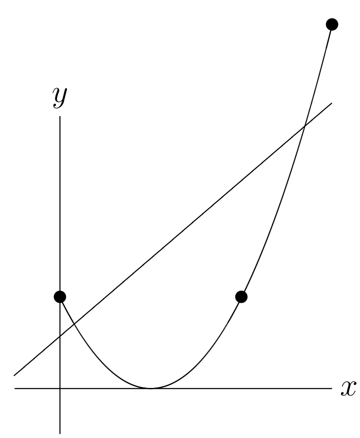
Example: For the same points as above, use least squares to fit a parabola.
<u>answer:</u> A parabola has the formula $y = ax^2 + bx + c$.
Squared error $= E(a,b,c) = \sum (y_i - (ax_i^2 + bx_i + c))^2$.
The least squares fit minimizes $E \ \Rightarrow\ \frac{\partial E}{\partial a} = \frac{\partial E}{\partial b} = \frac{\partial E}{\partial c} = 0$:

$$\Rightarrow \quad \begin{aligned} a\sum x_i^4 + b\sum x_i^3 + c\sum x_i^2 &= \sum x_i^2 y_i \\ a\sum x_i^3 + b\sum x_i^2 + c\sum x_i &= \sum x_i y_i \\ a\sum x_i^2 + b\sum x_i + c\,n &= \sum y_i \end{aligned}$$

Plugging in our points:
$\sum x_i^4 = 97, \quad \sum x_i^3 = 35, \quad \sum x_i^2 = 13, \quad \sum x_i = 5, \quad n = 3.$
$\sum x_i^2 y_i = 40, \quad \sum x_i y_i = 14, \quad \sum y_i = 6.$

$$\Rightarrow \quad \begin{aligned} 97a + 35b + 13c &= 40 \\ 35a + 13b + 5c &= 14 \\ 13a + 5b + 3c &= 6. \end{aligned}$$

Solving $\Rightarrow a = 1, \quad b = -2, \quad c = 1.$
The least squares parabola has equation $y = x^2 - 2x + 1$.
Note: for 3 points the fit is perfect.



Least squares line and parabola