18.02A Topic 29: Second derivative test, Lagrange multipliers.
Read: TB: 19.7, 19.8

Review: $\nabla w = 0 \Rightarrow$ critical point

Second derivative test for critical point (x_0, y_0)
Let $(w_{xx})_0 = A$, $(w_{xy})_0 = B$, $(w_{yy})_0 = C$
If $AC - B^2 > 0$ then $A > 0 \Rightarrow$ minimum, $A < 0 \Rightarrow$ maximum.
If $AC - B^2 < 0$ then saddle.
If $AC - B^2 = 0$ then test fails.

Example: $w = x^3 - 3xy + y^3$
$w_x = 3x^2 - 3y$, $w_y = -3x + 3y^2 \Rightarrow \nabla w = \langle 3x^2 - 3y, -3x + 3y^2 \rangle$
Critical points: $3x^2 - 3y = 0 \Rightarrow y = x^2$.
Substitute this into $-3x + 3y^2 = 0 \Rightarrow x^4 - x = 0 \Rightarrow x = 0, 1$
$\Rightarrow$ critical points are $(0, 0)$, $(1, 1)$.
$w_{xx} = 6x$, $w_{xy} = -3$, $w_{yy} = 6y \Rightarrow AC - B^2 = 36xy - 9$
$\Rightarrow$ $(0, 0)$ is a saddle and $(1, 1)$ is a minimum.

Reasoning: Second order approximation: (at $(0, 0)$)

$$w - w_0 \approx \left.\frac{\partial w}{\partial x}\right|_0 x + \left.\frac{\partial w}{\partial y}\right|_0 y + \frac{1}{2}\left.\frac{\partial^2 w}{\partial x^2}\right|_0 x^2 + \left.\frac{\partial^2 w}{\partial x \partial y}\right|_0 xy + \frac{1}{2}\left.\frac{\partial^2 w}{\partial y^2}\right|_0 y^2.$$

More simply: $\Delta w \approx (w_x)_0 x + (w_y)_0 y + \frac{1}{2}(w_{xx})_0 x^2 + (w_{xy})_0 xy + \frac{1}{2}(w_{yy})_0 y^2$.
At a critical point this (since $w_x = 0, w_{xx} = A$ etc.) becomes
$\Delta w = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2)$.
Complete square: $\Delta w = A(x + \frac{B}{A}y)^2 + \frac{1}{A}(AC - B^2)y^2$
So, if $AC - B^2 > 0$ and $A > 0$ then $\Delta w > 0 \Rightarrow$ minimum.
If $AC - B^2 > 0$ and $A < 0$ then $\Delta w < 0 \Rightarrow$ maximum.
If $AC - B^2 < 0$ then Δw varies $\Rightarrow$ saddle.

Examples: (Use these to remember the rules.)
i) $z = x^2 + y^2$ (min. at (0,0)): $A = 2$, $B = 0$, $C = 2$
$\Rightarrow AC - B^2 = 4 > 0$ and $A > 0$.
ii) $z = -(x^2 + y^2)$ (max. at (0,0)): $A = -2$, $B = 0$, $C = -2$
$\Rightarrow AC - B^2 = 4 > 0$ and $A < 0$.
iii) $z = y^2 - x^2$ (saddle. at (0,0)): $A = -2$, $B = 0$, $C = 2$
$\Rightarrow AC - B^2 = -4 < 0$.
iv) $z = xy$ (saddle. at (0,0)): $A = 0$, $B = 1$, $C = 0 \Rightarrow AC - B^2 = -1 < 0$.

General example: $z = \dfrac{1}{2}(ax^2 + 2bxy + cy^2)$ (crit. pt. at (0,0)):
$A = a$, $B = b$, $C = c \Rightarrow AC - B^2 = ac - b^2 < 0$.

(continued)

Lagrange multipliers:
Problem: Minimize $w = f(x, y, z)$ constrained by $g(x, y, z) = c$.

Sphere example:
Minimize $w = y$ constrained to $x^2 + y^2 + z^2 = 1$.

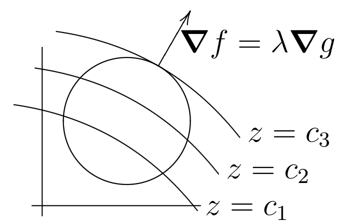
Example: Box: No top, sides double thick, bottom triple thick, volume = 3.
What's the smallest amount of cardboard you can use?
Dimensions: x, y, z.
Cardboard: $w = 4xz + 4yz + 3xy$.
Constraint $V = xyz = 3$.

Lagrange solution: Critical point $\nabla f = \lambda \nabla g$, constraint $g(x, y, z) = c$.

Sphere example: $\nabla f = \langle 0, 1, 0\rangle$, $\nabla g = \langle 2x, 2y, 2z\rangle$
$\nabla f = \lambda \nabla g \Rightarrow \langle 0, 1, 0\rangle = \lambda\langle 2x, 2y, 2z\rangle \Rightarrow x = z = 0$.
Constraint $\Rightarrow y = \pm 1$. (Gives min and max).

Box example: $\nabla f = \langle 4z + 3y, 4z + 3x, 4x + 4y\rangle$, $\nabla V = \langle yz, xz, xy\rangle$
Lagrange: $\langle 4z + 3y, 4z + 3x, 4x + 4y\rangle = \lambda\langle yz, xz, xy\rangle, \quad xyz = 3$
Solve symmetrically:
$\frac{4z+3y}{yz} = \lambda \quad \frac{4z+3x}{xz} = \lambda, \quad \frac{4x+4y}{xy} = \lambda, \quad xyz = 3$
$\Rightarrow \frac{4}{y} + \frac{3}{z} = \frac{4}{x} + \frac{3}{z} = \frac{4}{y} + \frac{4}{x}$
$\Rightarrow \frac{4}{y} = \frac{4}{x} \Rightarrow x = y$ and $\frac{3}{z} = \frac{4}{x} \Rightarrow z = \frac{3}{4}x$
$xyz = 3 \Rightarrow \frac{3}{4}x^3 = 3 \Rightarrow x = 4^{1/3}$
Answer: $x = 4^{1/3}, \quad y = 4^{1/3}, \quad z = 3 \cdot 4^{-2/3}, \quad w = 9 \cdot 4^{2/3}$.

Reason for Lagrange (using two dimensional picture)
Problem: minimize $w = f(x, y)$ subject to constraint $g(x, y) = c$.
Follow the level curves of f, the last one to touch $g = c$ is the maximum (or minimum) and it is tangent $\Rightarrow$ gradients are parallel.

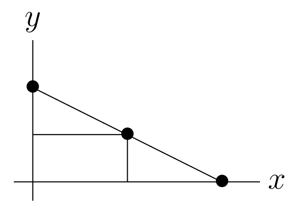


Reason for Lagrange (using analysis)
Constaint $g(x, y, z) = c$ is a level surface with normal ∇g.
Suppose P_0 is a minimum for f on the surface.
Let $\mathbf{r}(t)$ be any curve on the surface with $\mathbf{r}(0) = P_0$.
$\Rightarrow h(t) = f(\mathbf{r}(t))$ has a minimum at $t = 0$.
Taking a derivative: $h'(t) = \nabla f|_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)$.
$\Rightarrow 0 = h'(0) = \nabla f|_{P_0} \cdot \mathbf{r}'(0)$.
$\Rightarrow \nabla f|_{P_0}$ is perpendicular to any curve on the surface through P_0.
$\Rightarrow \nabla f|_{P_0}$ is perpendicular to the surface.
$\Rightarrow \nabla f|_{P_0}$ is parallel to $\nabla g|_{P_0}$.

(continued)

Example: (checking the boundary)
A rectangle in the plane is placed in the first quadrant so that one corner Q is at the origin and the two sides adjacent to Q are on the axes. The corner P opposite Q is on the curve $x + 2y = 1$. Using Lagrange multipliers find for which point P the rectangle has maximum area. Say how you know this point gives the maximum.

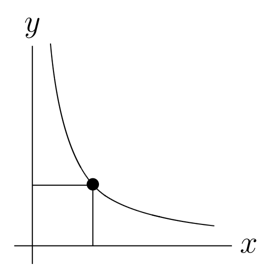
<u>**answer:**</u> We need some names.
Let $g(x, y) = x + 2y = 1$ = constraint and $f(x, y) = xy$ = area.
Gradients: $\nabla g = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, $\nabla f = y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$.
Lagrange multipliers: $\Rightarrow$ $y = \lambda$
$x = 2\lambda$
$x + 2y = 1$



The first two equations $\Rightarrow$ $x = 2y$;
Combine this with the third equation $\Rightarrow$ $4y = 1$.
$\Rightarrow$ $y = 1/4$, $x = 1/2$ $\Rightarrow$ $P = (1/2, 1/4)$.
We know this is a maximum because the maximum occurs either at a critical point or on the boundary. In this case the boundary points are on the axes which gives a rectangle with area = 0.

Example: (boundary at ∞)
A rectangle in the plane is placed in the first quadrant so that one corner Q is at the origin and the two sides adjacent to Q are on the axes. The corner P opposite Q is on the curve $xy = 1$. Using Lagrange multipliers find for which point P the rectangle has minimum perimeter. Say how you know this point gives the minimum.

<u>**answer:**</u> Let $g(x, y) = xy = 1$ = constraint and $f(x, y) = 2x + 2y$ = perimeter.
Gradients: $\nabla g = y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, $\nabla f = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.
Lagrange multipliers: $\Rightarrow$ $2 = \lambda y$
$2 = \lambda x$
$xy = 1$



The first two equations $\Rightarrow$ $x = y$;
Combine this with the third equation $\Rightarrow$ $x^2 = 1$.
$\Rightarrow$ $x = 1$, $x = 1$ $\Rightarrow$ $P = (1, 1)$.
We know this is a minimum because the minimum occurs either at a critical point or on the boundary. In this case the boundary points are infinitely far out on the axes which gives a rectangle with perimeter = ∞.