18.01A Topic 3: Indeterminate forms, L'Hospital's rule, growth rate of functions.
Read: TB: 12.2, 12.3 (examples 1-3, remark 1).

Warning: Pay attention to remark on p. 407 about 'The L'Hospital Habit'.

Examples: (warmup)

1. $\lim_{x\to 0} \frac{5x+1}{2x+3} = \frac{1}{3}$.
2. $\lim_{x\to 0} \frac{\sin x}{2x+3} = \frac{0}{3}$.

Examples: Indeterminant form

1. $\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$.

L'Hospital's Theorem for indeterminant form $\frac{0}{0}$.

If $f(x), g(x)$ are differentiable and $f(a) = g(a) = 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Examples:

1. $\lim_{x\to 0} \frac{\sin x}{x} \overset{\frac{0}{0}}{=} \lim \frac{\cos x}{1} = 1$. (This particular example is circular, since we need this limit to compute the derivative of $\sin x$.)
2. $\lim_{x\to 1} \frac{x-1}{3x-3} \overset{\frac{0}{0}}{=} \lim \frac{1}{3} = \frac{1}{3}$ (also can use algebra).

'Proof' of l'hospital:

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(f(x) - f(a))/(x-a)}{(g(x) - g(a))/(x-a)} = \frac{f'(a)}{g'(a)}$.

Note: the first equality is only true because $f(a) = 0$ and $g(a) = 0$.

Examples:

3. $\lim_{x\to 0} \frac{\sin^2 x}{x^2} \overset{\frac{0}{0}}{=} \lim \frac{2\sin x \cos x}{2x} \overset{\frac{0}{0}}{=} \lim \frac{2(\cos^2 x - \sin^2 x)}{2x} = 1$.

or use algebra: $(\lim_{x\to 0} \frac{\sin x}{x})^2 = 1^2 = 1$.

Other indeterminant forms: $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞.
L'Hospital only works for $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
For the others must manipulate to $0/0$ or ∞/∞.

(continued)

Examples:

1. $\lim\limits_{x\to 0} \dfrac{\ln x}{1/x} \overset{-\infty/\infty}{=} \lim \dfrac{1/x}{-1/x^2} = \lim \dfrac{-x^2}{x} = 0.$
2. $\lim\limits_{x\to 0^+} x\ln x \overset{0\cdot(-\infty)}{=}$ (must change to allowable form) $= \lim \frac{\ln x}{1/x} = 0.$

Examples: (when not to use)

1. Not indeterminant form: $\lim\limits_{x\to 0} \dfrac{x^2}{2x^2+5} = \dfrac{0}{5} = 0.$
2. $\lim\limits_{x\to 0} \dfrac{2x^2+5}{x^2} = \dfrac{5}{0}$ NO LIMIT.
3. $\lim\limits_{x\to 0} \dfrac{\sin x}{x^2} \overset{\frac{0}{0}}{=} \lim \dfrac{\cos x}{2x} = \dfrac{1}{0}$ NO LIMIT.

L'Hospital also works for limits as $x \to \infty$.

proof: Suppose $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$. Let $u = 1/x$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{u\to 0+} \frac{f(1/u)}{g(1/u)} \overset{\frac{0}{0}}{=} \lim \frac{f'(1/u)(-1/u^2)}{g'(1/u)(-1/u^2)} = \lim \frac{f'(1/u)}{g'(1/u)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Examples:

1. $\lim\limits_{x\to\infty} \dfrac{x}{e^x} \overset{\infty/\infty}{=} \lim \dfrac{1}{e^x} = 0.$
2. $\lim\limits_{x\to 0+} x\ln x = \lim\limits_{u\to\infty} \dfrac{1}{u}\ln(1/u) = \lim \dfrac{-\ln u}{u} \overset{\infty/\infty}{=} \lim \dfrac{-1/u}{1} = 0.$

Examples: (polynomials)

1. $\lim\limits_{x\to\infty} \dfrac{x^3+x^2+1}{(2x+1)^3}$. Either use L'Hospital 3 times or divide by x^3.

Dividing: $\lim \dfrac{1+1/x+1/x^3}{(2+1/x)^3} = \dfrac{1}{8}.$

Examples:

1. $\lim\limits_{x\to\infty} x^{1/x} = \infty^0.$

Set $y = \lim\limits_{x\to\infty} x^{1/x} \;\Rightarrow\; \ln y = \lim \dfrac{1}{x}\ln x = 0 \;\Rightarrow\; y = 1.$