18.02A Topic 31: Double and iterated integrals.
Read: TB: 20.1, 20.2

Double integral –geometric
Iterated integral –analytic

Iterated integrals

Example: $\int_0^1 \int_0^2 xy\,dx\,dy$

Inner integral: $\int_0^2 xy\,dx = \left.\frac{x^2}{2}y\right|_0^2 = 2y$

Outer integral: $\int_0^1 2y\,dy = \left. y^2\right|_0^1 = 1.$

Example: $\int_0^1 \int_{x^2}^x (2x+2y)\,dy\,dx$

Inner integral: $\int_{x^2}^x (2x+2y)\,dy = \left. 2xy + y^2\right|_{x^2}^x = 2x^2 + x^2 - (2x^3 + x^4) = 3x^2 - 2x^3 - x^4$

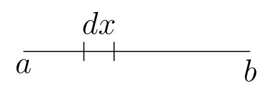
Outer integral: $\int_0^1 3x^2 - 2x^3 - x^4\,dx = \left. x^3 - \frac{x^4}{2} - \frac{x^5}{5}\right|_0^1 = 1 - \frac{1}{2} - \frac{1}{5}.$

Double integrals Mass example
1 dimension:

density $= \delta(x)$

$dm = \delta(x)\,dx$

$M = \int_a^b \delta(x)\,dx$



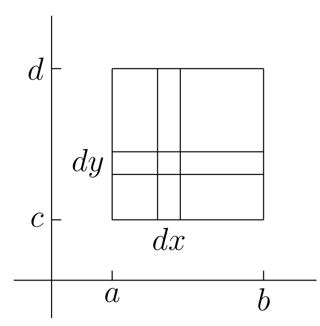
2 dimensions:

density $= \delta(x, y)$

$dm = \delta(x,y)\,dA = \delta(x,y)\,dx\,dy$

$M = \int\int_R \delta(x,y)\,dA = \int\int_R \delta(x,y)\,dx\,dy$

= Double integral



To compute: mass (horiz.) strip (fix y) $= \int_a^b \delta(x,y)\,dx$

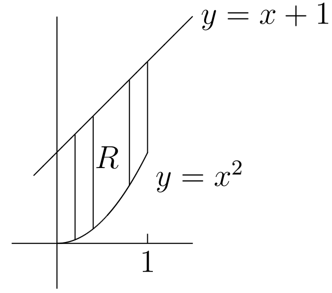
M = "sum" mass of strips $= \int_c^d \int_a^b \delta(x,y)\,dx\,dy$

= iterated integral.

(continued)

Limits of integration

Example: Find the mass of region R bounded by
$y = x + 1$, $y = x^2$, $x = 0$, $x = 1$, density $= \delta(x, y) = xy$
x is between 0 and 1.
As x moves the vertical lines sweep out R.
Fix x then y runs from x^2 to $x + 1$.



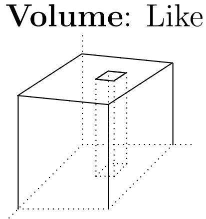
$$\Rightarrow\ M = \int\int_R \delta(x, y)\, dA = \int_{x=0}^{1}\int_{y=x^2}^{x+1} xy\, dy\, dx$$

Inner: $\displaystyle\int_{x^2}^{x+1} xy\, dy = x\frac{y^2}{2}\bigg|_{x^2}^{x+1} = \frac{x(x+1)^2}{2} - \frac{x^5}{2} = \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}$

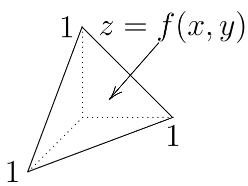
Outer: $\displaystyle\int_0^1 \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}\, dx = \frac{x^4}{8} + \frac{x^3}{3} + \frac{x^2}{4} - \frac{x^6}{12}\bigg|_0^1 = \frac{1}{8} + \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{5}{8}.$

Note: The syntax $y = x^2$ in limits is redundant but useful. We know it must be y because of the dy matching the integral sign...

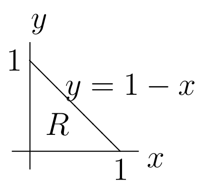
Volume: Like area: volume = 'sum' of rectangular boxes.



$dV = z\, dA$



Tetrahedron



Region R

Example: Volume of tetrahedron (see above pictures)
Surface: $z = 1 - x - y$.
Limits: x: 0 to 1; y: 0 to $1 - x$.

$$\Rightarrow\ V = \int_{x=0}^{1}\int_{y=0}^{1-x} 1 - x - y\, dy\, dx.$$

Inner: $\displaystyle\int_{y=0}^{1-x} 1 - x - y\, dy = y - xy - \frac{y^2}{2}\bigg|_0^{1-x} = 1 - x - x + x^2 - \frac{1}{2} + x - \frac{x^2}{2}$

Outer: $\displaystyle\int_0^1 \frac{1}{2} - x + \frac{x^2}{2}\, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$

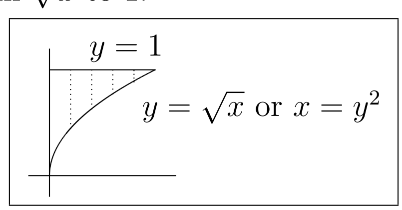
Changing order of integration

Example: $\displaystyle\int_0^1\int_{\sqrt{x}}^1 \frac{e^y}{y}\, dy\, dx.$ –Inner integral is too hard –so change order:

1) Find limits for region R: x from 0 to 1; fix x: y from $\sqrt{x}$ to 1.

2) Reverse limits: y from 0 to 1; fix y: x from 0 to y^2.

3) Compute integral: $\displaystyle\int_{y=0}^{1}\int_{x=0}^{y^2} \frac{e^y}{y}\, dx\, dy$



Inner: $\displaystyle x\frac{e^y}{y}\bigg|_0^{y^2} = ye^y \Rightarrow$ Outer: $\displaystyle\int_0^1 ye^y = ye^y - e^y\big|_0^1 = 1.$