18.02A Topic 31: Double and iterated integrals.
Read: TB: 20.1, 20.2

Double integral –geometric
Iterated integral –analytic

Iterated integrals

Example: \( \int_0^1 \int_0^2 xy \, dx \, dy \)

Inner integral: \( \int_0^2 xy \, dx = \frac{x^2}{2} \bigg|_0^2 y = 2y \)
Outer integral: \( \int_0^1 2y \, dy = y^2 \bigg|_0^1 = 1 \).

Example: \( \int_0^1 \int_x^2 (2x + 2y) \, dy \, dx \)

Inner integral: \( \int_x^2 (2x + 2y) \, dy = 2xy + y^2 \bigg|_x^2 = 2x^2 + x^2 - (2x^3 + x^4) = 3x^2 - 2x^3 - x^4 \)
Outer integral: \( \int_0^1 3x^2 - 2x^3 - x^4 \, dx = x^3 - \frac{x^4}{2} - \frac{x^5}{5} \bigg|_0^1 = 1 - \frac{1}{2} - \frac{1}{5} \).

Double integrals Mass example

1 dimension:
- density = \( \delta(x) \)
- \( dm = \delta(x) \, dx \)
- \( M = \int_a^b \delta(x) \, dx \)

2 dimensions:
- density = \( \delta(x, y) \)
- \( dm = \delta(x, y) \, dA = \delta(x, y) \, dx \, dy \)
- \( M = \int \int_R \delta(x, y) \, dA = \int \int_R \delta(x, y) \, dx \, dy \)
- = Double integral

To compute: mass (horiz.) strip (fix \( y \)) = \( \int_a^b \delta(x, y) \, dx \)

\( M = \) "sum" mass of strips = \( \int_c^d \int_a^b \delta(x, y) \, dx \, dy \)
- = iterated integral.

(continued)
Limits of integration

**Example:** Find the mass of region $R$ bounded by $y = x + 1$, $y = x^2$, $x = 0$, $x = 1$, density $\delta(x, y) = xy$

$x$ is between 0 and 1.

As $x$ moves the vertical lines sweep out $R$.

Fix $x$ then $y$ runs from $x^2$ to $x + 1$.

$$M = \int \int_R \delta(x, y) \, dA = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x+1} xy \, dy \, dx$$

Inner:
$$\int_{x^2}^{x+1} xy \, dy = \frac{y^2}{2} \bigg|_{x^2}^{x+1} = \frac{x(x + 1)^2}{2} - \frac{x^5}{2} = \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}$$

Outer:
$$\int_0^1 \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2} \, dx = \frac{x^4}{8} + \frac{x^3}{3} + \frac{x^2}{4} - \frac{x^6}{12} \bigg|_0^1 = \frac{1}{8} + \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{5}{8}.$$ 

**Note:** The syntax $y = x^2$ in limits is redundant but useful. We know it must be $y$ because of the $dy$ matching the integral sign...

**Volume:** Like area: volume = 'sum' of rectangular boxes.

**Example:** Volume of tetrahedron (see above pictures)

Surface: $z = 1 - x - y$.

Limits: $x$: 0 to 1; $y$: 0 to 1 - $x$.

$$\Rightarrow V = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 1 - x - y \, dy \, dx.$$ 

Inner:
$$\int_{y=0}^{1-x} 1 - x - y \, dy = y - xy - \frac{y^2}{2} \bigg|_0^{1-x} = 1 - x - x^2 - \frac{1}{2} + x - \frac{x^2}{2}$$

Outer:
$$\int_0^1 \frac{1}{2} - x + \frac{x^2}{2} \, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$$ 

**Changing order of integration**

**Example:** $\int_0^1 \int_0^{\sqrt{x}} \frac{e^y}{y} \, dy \, dx$. –Inner integral is too hard –so change order:

1) Find limits for region $R$: $x$ from 0 to 1; fix $x$: $y$ from $\sqrt{x}$ to 1.

2) Reverse limits: $y$ from 0 to 1; fix $y$: $x$ from 0 to $y^2$.

3) Compute integral: $\int_{y=0}^{y^2} \int_{x=0}^{y^2} \frac{e^y}{y} \, dx \, dy$

Inner:
$$\frac{e^y}{y} \bigg|_0^{y^2} = ye^y \Rightarrow \text{Outer: } \int_0^1 ye^y = ye^y - e^y \bigg|_0^1 = 1.$$