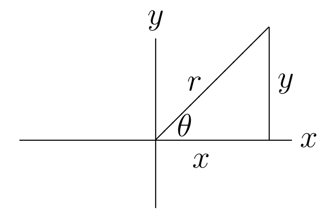
18.02A Topic 32: Polar coordinates, double integrals in polar coordinates.
Read: TB: 16.1, 20.4

Polar Coordinates
Definition:
$x = r\cos\theta,\ y = r\sin\theta.$
$r = \sqrt{x^2+y^2}$, "$\theta = \tan^{-1}(y/x)$."
(θ is tricky, $\tan^{-1}$ is in quotes to indicate you need to pick the correct quadrant. Use the picture –see example below.)



Standard x, y, r, θ triangle

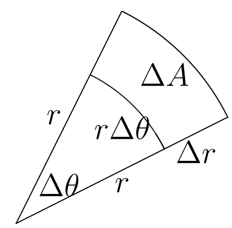
Example: (more than one way to represent any point) (The x,y coordinates are at the top of each column and various r,θ representations are below.)

(x,y)	$(1,0)$	$(0,1)$	$(2,0)$	$(1,1)$	$(-1,1)$	$(-1,-1)$	$(0,0)$
(r,θ)	$(1,0)$	$(1,\pi/2)$	$(2,0)$	$(\sqrt{2},\pi/4)$	$(\sqrt{2},3\pi/4)$	$(\sqrt{2},5\pi/4)$	$(0,\pi/2)$
(r,θ)	$(1,2\pi)$			$(\sqrt{2},9\pi/4)$		$(-\sqrt{2},\pi/4)$	$(0,-7.2)$
(r,θ)	$(1,4\pi)$						

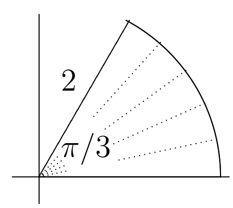
Double integral: $\int\int_R f(x,y)\,dA$

Write dA in polar coordiantes

$\Delta A \approx r\Delta\theta\Delta r$
$\Rightarrow\ dA = r\,d\theta\,dr = r\,dr\,d\theta$



Example: Find the mass of the region R shown if it has density $\delta(x,y) = xy$
In polar coordinates: $\delta = r^2\cos\theta\sin\theta$.
Limits of integration: radial lines sweep out R
$\Rightarrow\ \theta$: 0 to $\pi/3$; fix $\theta\ \Rightarrow\ r$: 0 to 2.
$\Rightarrow\ \int\int_R \delta(x,y)\,dA = \int_{\theta=0}^{\pi/3}\int_{r=0}^{2} r^2\cos\theta\sin\theta\, r\,d\theta\,dr$
Inner: $\int_0^2 r^3\cos\theta\sin\theta\,dr = \frac{r^4}{4}\cos\theta\sin\theta\Big|_0^2 = 4\cos\theta\sin\theta$
Outer: $\int_0^{\pi/3} 4\cos\theta\sin\theta\,d\theta = 2\sin^2\theta\Big|_0^{\pi/3} = \frac{3}{2}$.

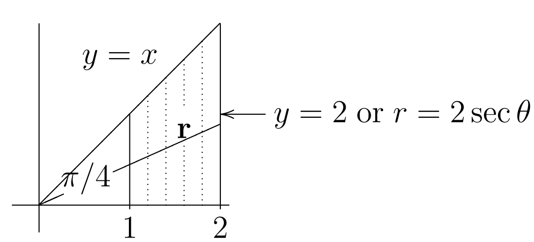


Example: $I = \int_1^2\int_0^x \frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx$
Draw the region.
In polar coordinates:
θ: 0 to $\pi/4$; fix $\theta\ \Rightarrow\ r$: $\sec\theta$ to $2\sec\theta$.

$$\Rightarrow\ I = \int_{\theta=0}^{\pi/4}\int_{r=\sec\theta}^{2\sec\theta}\frac{1}{r^3}r\,dr\,d\theta.$$

Inner: $$\int_{\sec\theta}^{2\sec\theta}\frac{1}{r^2}\,dr = -\frac{1}{r}\Big|_{\sec\theta}^{2\sec\theta} = \frac{1}{2}\cos\theta.$$

Outer: $$\int_0^{\pi/4}\frac{1}{2}\cos\theta\,d\theta = \frac{1}{2}\sin\theta\Big|_0^{\pi/4} = \frac{\sqrt{2}}{4}.$$



(continued)

Example: Find the volume of the region above the xy-plane and below the graph of $z = 1 - x^2 - y^2$.

You should draw a picture of this.

In polar coordinates we have $z = 1 - r^2$ and we want the volume under the graph and above the inside of the unit disk.

$\Rightarrow$ volume $= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r dr\, d\theta$.

Inner integral: $\int_0^1 (1 - r^2)\, r dr = \frac{1}{2} - frac14 = \frac{1}{4}$.

Outer integral: $\int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}$ = volume.