18.02A Topic 33: Change of variable.
Read: SN: CV

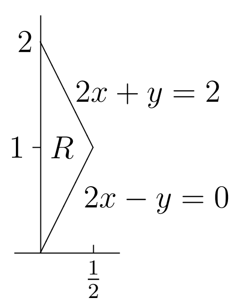
18.01 example: Compute $I = \int x\sqrt{1-x^2}\,dx$.
Direct substitution: $u = 1 - x^2, \quad du = -2x\,dx \to x\,dx = -\frac{1}{2}\,du$
$\Rightarrow \; I = -\frac{1}{2}\int \sqrt{u}\,du$.
Inverse substitution: $x = \sin u, \quad dx = \cos u\,du \;\Rightarrow\; I = \int \sin u \cos^2 u\,du$.

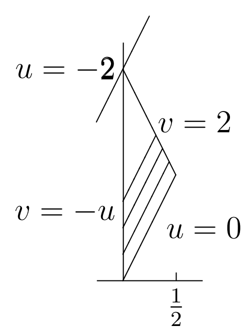
Double integration: $\int\int_R f(x,y)\,dA$.
Direct substitution: $u = u(x,y)$, $v = v(x,y)$.
Inverse substitution: $x = x(u,v)$, $y = y(u,v)$.
Example: $x = r\cos\theta$, $y = r\sin\theta$.

Finding $dA = dx\,dy$:
Jacobian $= \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}, \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$.
$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}\,du\,dv$.
(Reason soon)

Change of variable for $\int\int_R f(x,y)\,dx\,dy$:
(A) Change $f(x,y)$ to u, v coordinates.
(B) Express dA in terms of $du\,dv$.
(C) Find u, v limits on R.

Example: Given the region R shown, compute $I = \int\int_R (4x^2 - y^2)^4\,dy\,dx$.
Change of variable: $u = 2x - y$, $v = 2x + y$.
(A) $4x^2 - y^2 = uv \;\Rightarrow\; f(x,y) = u^4v^4$.
(B) $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$.
$\Rightarrow\; \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4}. \;\Rightarrow\; dx\,dy = \frac{1}{4}du\,dv$.
(C) Limits:
$x = 0 \;\Rightarrow\; u = -y,\, v = y \;\Rightarrow\; v = -u$.
u: -2 to 0; fix $u \;\Rightarrow\; v$: $-u$ to 2.
$$I = \int_{u=-2}^{0}\int_{v=-u}^{2} u^4v^4\,\frac{dv\,du}{4}.$$
(Easy to compute.)





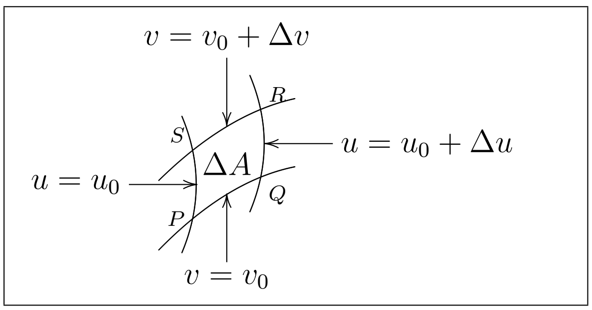
(continued)

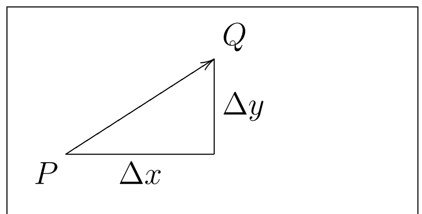
Reason for change of variables formula:
Draw 'grid' lines $u = c_1$, $v = c_2$ (in xy-plane).
ΔA = area $PQRS \approx$ parallelogram.
In a moment we will see that $\overrightarrow{\mathbf{PQ}} \approx \langle x_u \Delta u, y_u \Delta u\rangle$ and $\overrightarrow{\mathbf{PS}} \approx \langle x_v \Delta v, y_v \Delta v\rangle$.
Using this we have:

$$\Delta A \approx \pm \begin{vmatrix} x_u\Delta u & x_v \Delta v \\ y_u \Delta u & y_v \Delta v \end{vmatrix} = \pm \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \Delta u \Delta v = \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v.$$

$\Rightarrow \; dA = \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv.$ ■

Now we show $\overrightarrow{\mathbf{PQ}} \approx \langle x_u \Delta u, y_u \Delta u\rangle$.
Let $\overrightarrow{\mathbf{PQ}} = \langle \Delta x, \Delta y\rangle$ (see picture).
The approximation formula says: $\Delta x \approx x_u \Delta u + x_v \Delta v$ and $\Delta y \approx y_u \Delta u + y_v \Delta v$.
Going from P to Q we have $\Delta v = 0 \;\Rightarrow\; \Delta x \approx x_u \Delta u$ and $\Delta y \approx y_u \Delta u$.
$\Rightarrow \; \overrightarrow{\mathbf{PQ}} \approx \langle x_u \Delta u, y_u \Delta u\rangle$. ■
Likewise for $\overrightarrow{\mathbf{PS}} \approx \langle x_v \Delta v, y_v \Delta v\rangle$.





Detail from picture at left

Remark on the chain rule:

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1 \quad \text{or} \quad \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \cdot \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = I$$

$$\Leftrightarrow \; 1 = \frac{\partial x}{\partial x} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} \text{ etc.}$$