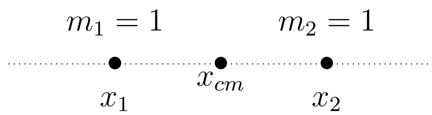
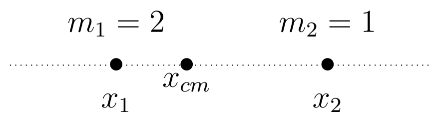
18.02A Topic 34: Applications of double integration.
Read: TB: 20.3

Center of Mass

In one dimension:

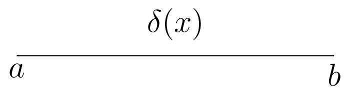


$x_{cm} = \frac{x_1+x_2}{2}$.



$x_{cm} = \frac{2x_1+x_2}{3}$.

In general, x_{cm} = weighted average of position $= \frac{\sum m_i x_i}{\sum m_i}$.
For continuous density:

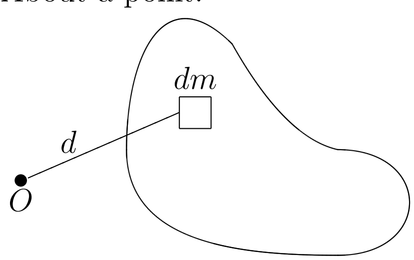


$M = \int_a^b \delta(x)\,dx, \quad x_{cm} = \frac{\int x\,dm}{M} = \frac{\int x\delta(x)\,dx}{\int \delta(x)\,dx}$.

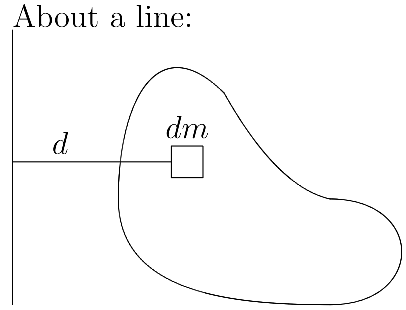
In 2 dimensions:
$M = \int\int_R \delta(x,y)\,dA, \quad x_{cm} = \int\int_R x\delta(x,y)\,dA/M, \quad y_{cm} = \int\int_R y\delta(x,y)\,dA/M$

Moment of inertia: $I = \int\int_R d^2\,dm = \int\int_R d^2\delta(x,y)\,dA$.

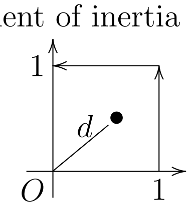
About a point:



About a line:



Example: $\delta = xy$; Find mass, center of mass and moment of inertia about O.

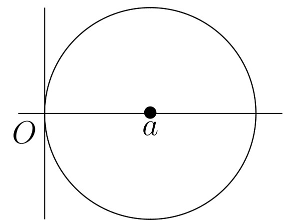


$M = \int\int_R \delta\,dA = \int_0^1\int_0^1 xy\,dx\,dy = \frac{1}{4}$.
$x_{cm} = \frac{1}{M}\int\int x\delta\,dA = \frac{1}{M}\int_0^1\int_0^1 x^2y\,dy\,dx$.
Inner (not including $\frac{1}{M}$): $\int_0^1 x^2y\,dy = \left.\frac{x^2y^2}{2}\right|_0^1 = \frac{x^2}{2}$.
Outer: $\int_0^1 \frac{x^2}{2}\,dx = \left.\frac{x^3}{6}\right|_0^1 = \frac{1}{6}$.
$\Rightarrow\ x_{cm} = \frac{4}{6} = \frac{2}{3}$. Symmetry $\Rightarrow\ y_{cm} = \frac{2}{3}$.
$I = \int\int_R r^2\delta\,dA = \int_0^1\int_0^1 (x^2+y^2)xy\,dx\,dy = \frac{1}{4}$.

(continued)

Example: Disk of radius a with center at $(a,0)$, $\delta(x,y)=1$.
Find moment of inertia about O.
$I=\int\int_R r^2\delta\,dA=\int\int_R r^2\,dA$.
In polar coords: boundary circle $= r=2a\cos\theta$; $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.
Limits: θ: $-\frac{\pi}{2}$ to $\frac{\pi}{2}$; fix θ $\Rightarrow$ r: 0 to $2a\cos\theta$.
$\Rightarrow$ $I=\int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r^2 r\,dr\,d\theta$.
Inner: $\left.\frac{r^4}{4}\right|_0^{2a\cos\theta}=4a^4\cos^4\theta$.
Outer:

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}4a^4\cos^4\theta\,d\theta &= a^4\int_{-\pi/2}^{\pi/2}\tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\\ &= a^4\left(\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right|_{-\pi/2}^{\pi/2}\\ &= a^4\tfrac{3}{2}\pi\\ &= M\tfrac{3}{2}a^2.\end{aligned}$$

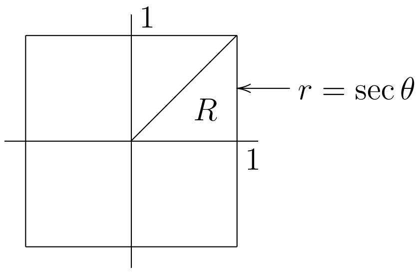


(Note: this agrees with the parallel axis theorem.)

Average Value

The average value of $f(x,y)$ with respect to area on a region R is $\displaystyle\frac{1}{\text{area } R}\int\int_R f(x,y)\,dA$.

Example: What's the average distance of a point in a square from the center?
answer: We center the square on the origin. By symmetry this is the same as the average distance from the origin of the triangular region R shown in the picture. In polar coordinates the distance is r and the area of the triangle is $\frac{1}{2}$ $\Rightarrow$ average distance $=\displaystyle\frac{1}{1/2}\int_0^{\pi/4}\int_0^{\sec\theta} r\,r\,dr\,d\theta=2\int_0^{\pi/4}\frac{\sec^3\theta}{3}\,d\theta=\tfrac{1}{3}(\sqrt{2}+\ln(\sqrt{2}+1))$.



Note: x_{cm} is the average value of x with respect to mass.
The geometric center has coordinates given by the average value of x and y with respect to area, i.e., the center of mass when $\delta=1$.