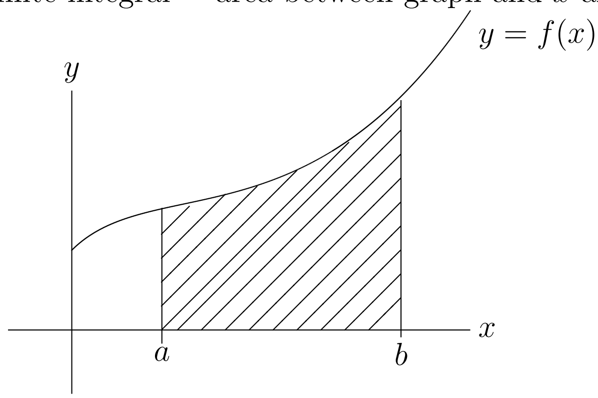
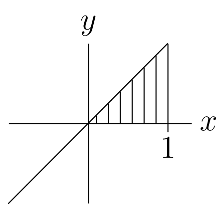
18.01A Topic 4: Definite integral; summation notation, first fund. theorem, properties.
Read: TB: 6.3 through formula (4); skip proofs; 6.4, 6.5, 6.6.

Definition: Definite integral = area between graph and x-axis.

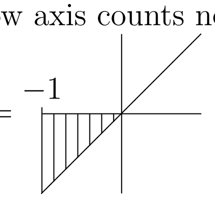
I.e. $\int_a^b f(x)\,dx =$ 

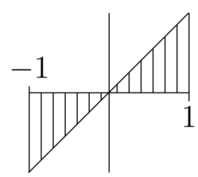
Examples: $\displaystyle\int_0^1 x\,dx =$ 

$= \dfrac{1}{2}.$

Dummy variables: The variable used in integration can be any letter.
$\Rightarrow \displaystyle\int_0^1 x\,dx = \int_0^1 u\,du = \int_0^1 t\,dt.$

Area below axis counts negative:

$\displaystyle\int_{-1}^0 x\,dx =$ 

$= -\dfrac{1}{2}.$ $\displaystyle\int_{-1}^1 x\,dx =$ 

$= 0.$

Summation notation:
We will be dealing with sums with many terms with a pattern.
Examples: 1. $1 + 2 + 3 + \ldots + 999 + 1000.$
2. $1 + 1/2 + 1/3 + \cdots + 1/999 + 1/1000.$
3. $1 + 2 + \ldots + N.$
The ellipsis indicates we didn't write down every term. Often this is okay since we can see the pattern. But, this is not always clear. One way to be fully specific and to be more compact is summation notation.

Examples: 1. $\displaystyle\sum_{n=1}^{1000} n = 1 + 2 + \ldots + 1000.$

(continued)

2. $\displaystyle\sum_{n=1}^{1000} 1/n = 1/1 + 1/2 + \ldots + 1/1000.$

3. $\displaystyle\sum_{n=1}^{N} n = 1 + 2 + \ldots + N.$

The letter $\sum$ is the uppercase Greek letter sigma –for summation.

In the examples, the letter n is the **index** and the terms above and below the $\sum$ are the limits.

$\displaystyle\sum_{n=1}^{1000} n^2$ is read as 'the sum from $n = 1$ to 1000 of n^2'.

Example: Compute $\displaystyle\sum_{j=2}^{5} j^2$.

<u>**answer:**</u> $2^2 + 3^2 + 4^2 + 5^2 = 54$.

Example: Write $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + 100 \cdot 101$ in summation notation.

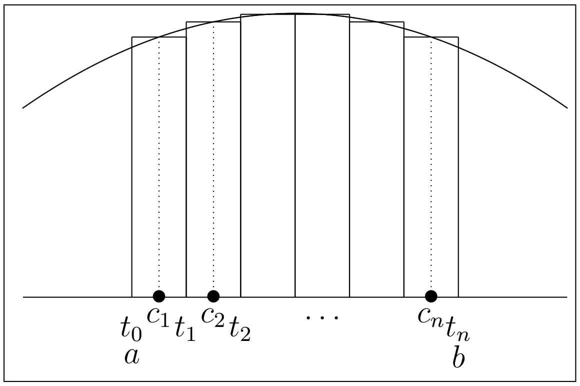
<u>**answer:**</u> $\displaystyle\sum_{k=1}^{100} k \cdot (k+1)$.

Example: Write the sum from $k = 7$ to 23 of $\sin(k\pi/100)$ in summation notation.

<u>**answer:**</u> $\displaystyle\sum_{k=7}^{23} \sin(k\pi/100)$.

Method of Exhaustion to compute $\displaystyle\int_a^b f(x)\,dx$:

(One of the main points of the class.)
Divide $[a, b]$ into n equal intervals
$\Rightarrow$ each has width $\Delta x = \frac{b-a}{n}$
Pick any c_i in the i-th interval.
i-th rectangle:
base $= \Delta x$, height $= f(c_i)$
Area under curve:
$\approx$ sum of area of rectangles
$= \sum_1^n f(c_i)\Delta x$
$=$ (definition) S_n
$=$ (definition) **Riemann Sum** .



As $n \to \infty$ the width $\Delta x \to 0$ and $S_n \to$area $= \displaystyle\int_a^b f(x)\,dx$.

(continued)

Theorem The above is independent of the choice of c_i.

Typical choices are: left endpoints, right endpoints, midpoint, biggest value, smallest value.
The Riemann sums are the called respectively the left, right, mid, upper and lower Riemann sum.

Example:

$$\int_0^1 x\,dx \approx \sum_{i=1}^{n}(\frac{i}{n})\cdot\frac{1}{n} \quad (\text{Here } \Delta x = \frac{1}{n},\ \text{right endpt} = \frac{i}{n}.)$$
$$= \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1+1/n^2}{2} = \frac{1}{2}.$$

Simmons computes $\int x\,dx$, $\int x^2\,dx$, $\int x^4\,dx$.
The computation relies on formulas for $\sum i$, $\sum i^2$, $\sum i^4$.

The definite integral is defined as an area. So far our only method of computing it is to use the rather tiring 'method of exhaustion'. There must be an easier way to compute integrals.

First Fundamental Theorem of Calculus:
If $f(x)$ is continuous and $F'(x) = f(x)$ then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = \ (\text{notation})\ \ F(x)|_a^b.}$$

I.e. Finding area ⇔ finding an anti-derivative. This is a BIG idea.

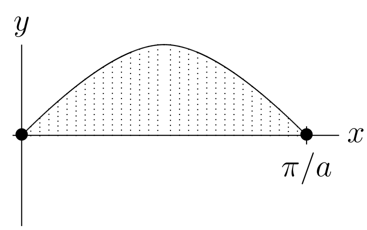
Note: When you see something called the 'Fundamental Theorem' you should assume it's important. In this case, it warrants a lot of attention and three proofs.

Examples:

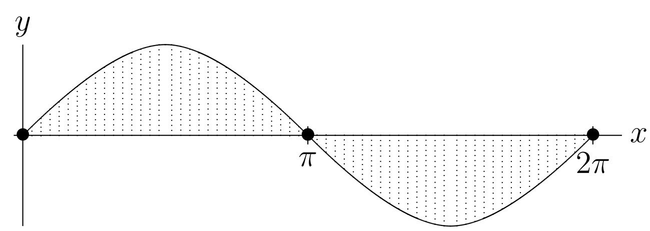
1. $\int_0^1 x^3\,dx = \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4}$ (draw your own picture).

2. $\int_0^{\pi/a} \sin ax\,dx = -\frac{1}{a}\cos ax\Big|_0^{\pi/a} = \frac{2}{a}$. (Note the picture shows the integral is positive –it's easy to mess up signs.)

3. $\int_0^{2\pi} \sin x\,dx = -\cos x|_0^{2\pi} = 0$.



$y = \sin ax$



$y = \sin x$

(continued)

4. $\int_1^2 \frac{1}{x}\,dx = \ln x|_1^2 = \ln 2.$

5. Given a rod of length 2 m with density $\delta(x) = 2 - (x-1)^2$ g/m, find the total mass of the rod.

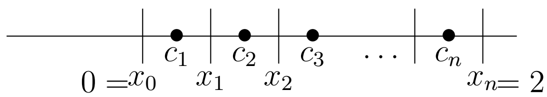
<u>**answer:**</u> Divide the rod into small segments of length Δx.

The mass of the i$^{\text{th}}$ segment $= \Delta m_i \approx \delta(c_i)\,\Delta x$.

$\Rightarrow$ Total mass $= \sum \Delta m_i \approx \sum \delta(c_i)\,\Delta x$.

In the limit the approximations are exact and the sum becomes an integral:

$\Rightarrow$ total mass $= \int_0^2 \delta(x)\,dx = \int_0^2 2 - (x-1)^2\,dx = 2x - (x-1)^3/3|_0^2 = 4 - 2/3.$



An important convention: So far we've always had $a < b$, the following will be quite useful.

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.}$$

Properties of definite integrals

1. $\displaystyle\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

2. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

4. $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

(All follow from the definition of integral as area.)

Example:

$$\begin{aligned}\int_1^2 3x^3 + 4x\,dx &= 3\int_1^2 x^3\,dx + 4\int_1^2 x\,dx \\ &= 3x^4/4|_1^2 + 4x^2/2|_1^2 \\ &= 17\tfrac{1}{4}.\end{aligned}$$

Example: (of property 2)

$$\begin{aligned}\int_{-1}^1 x^3\,dx &= \int_{-1}^0 x^3\,dx + \int_0^1 x^2\,dx \\ &= x^4/4|_{-1}^0 + x^4/4|_0^1 \\ &= -1/4 + 1/4 = 0.\end{aligned}$$

(continued)

Example: (of property 2)

$$\begin{aligned}\int_{-1}^{1} |x|\,dx &= \int_{-1}^{0} |x|\,dx + \int_0^1 |x|\,dx \\ &= \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx \\ &= -x^2/2\big|_{-1}^{0} + x^2/2\big|_0^1 \\ &= 1/2 + 1/2 = 1.\end{aligned}$$

Example: (of property 4)

$\int_{-1}^{1} x^3\,dx = x^4/4\big|_{-1}^{1} = 0.$

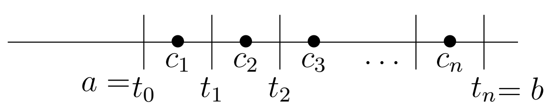
$\int_{-1}^{1} |x^3|\,dx = \int_{-1}^{0} -x^3\,dx + \int_0^1 x^3\,dx = 2/3.$

$\Rightarrow \left|\int_{-1}^{1} x^3\,dx\right| \le \int_{-1}^{1} |x^3|\,dx.$

For each of these examples you should be able to draw a picture and understand the algebraic manipulations in terms of areas.

Here are 3 promised proofs. I will not mention the MVT version again.

proof 1: (speed and distance)
Suppose position $= F(t)$.
$\Rightarrow$ speed $= F'(t) = f(t)$, net distance traveled $= F(b) - F(a)$.
Divide $[a, b]$ into n equal intervals, choose c_i as above.
Distance traveled in i-th interval $= \Delta s_i \approx f(c_i)\Delta t$.



Net distance $= \Delta s_1 + \Delta s_2 + \ldots + \Delta s_n = \displaystyle\sum_{i=1}^{n} \Delta s_i \approx \sum_{i=1}^{n} f(c_i)\Delta t.$

As $n \to \infty$ this approximation becomes exact and becomes Net distance $= \int_a^b f(t)\,dt$

Therefore: net distance $= F(b) - F(a) = \int_a^b f(t)\,dt.$ QED

proof 2: (Mean Value Theorem)

$$\begin{aligned}F(b) - F(a) &= \sum_{1}^{n} F(t_i) - F(t_{i-1}) \\ &= \sum F'(c_i)(t_i - t_{i-1}), \quad \text{where } c_i \text{ is from the MVT} \\ &= \sum f(c_i)\Delta t.\end{aligned}$$

As always, this sum $\to \displaystyle\int_a^b f(t)\,dt$ as $\Delta t \to 0$.

proof 3: See book §6.6