18.01A Topic 5: Second fundamental theorem, \( \ln x \) as an integral.
Read: SN: PI, FT.

First Fundamental Theorem: \( F' = f \Rightarrow \int_a^b f(x) \, dx = F(b) - F(a) \)

Questions: 1. Why \( dx \)? 2. Given \( f(x) \) does \( F(x) \) always exist?

Answer to question 1.
a) Riemann sum: Area \( \approx \sum f(c_i) \Delta x \to \int_a^b f(x) \, dx \)
In the limit the sum becomes an integral and the finite \( \Delta x \) becomes the infinitesimal \( dx \).
b) The \( dx \) helps with change of variable.

Example: Compute \( \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx \).
Let \( x = \sin u \)
\( \frac{dx}{du} = \cos u \Rightarrow dx = \cos u \, du, \quad x = 0 \Rightarrow u = 0, \quad x = 1 \Rightarrow u = \pi/2 \).
Substituting: \( \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2 u}} \cos u \, du = \int_0^{\pi/2} \frac{\cos u}{\cos u} \, du = \int_0^{\pi/2} du = \pi/2 \).

Answer to question 2. Yes →

Second Fundamental Theorem:
If \( f \) is continuous and \( F(x) = \int_a^x f(u) \, du \) then \( F'(x) = f(x) \).
I.e. \( f \) always has an antiderivative.

This is subtle: we have defined a NEW function \( F(x) \) using the definite integral.
Note we needed a dummy variable \( u \) for integration because \( x \) was already taken.

proof: \[ \Delta \text{area} = \int_x^{x+\Delta x} f(x) \, dx = \int_0^{\Delta x} f(x) \, dx - \int_0^x f(x) \, dx = F(x + \Delta x) - F(x) = \Delta F. \]
But, also \( \Delta \text{area} \approx f(x) \Delta x \Rightarrow \Delta F \approx f(x) \Delta x \) or \( \frac{\Delta F}{\Delta x} \approx f(x) \).
As \( \Delta x \to 0 \) this becomes exact: \( \frac{dF}{dx} = f(x) \). QED

More subtlety: For any continuous function there is an antiderivative. We might not know it in closed form but we can always write it as a definite integral with a variable limit. This is useful since Riemann sums let us compute it as accurately as we wish.

Examples: (Not elementary functions BUT they are functions.)
\( F(x) = \int_0^x e^{-t^2} \, dt \) statistics.
\( \text{Li}(x) = \int_2^x \frac{1}{\ln t} \, dt \) number theory.
\( \text{Si}(x) = \int_0^x \sin(t^2) \, dt \) optics.

(continued)
Natural logarithm as a definite integral: \( \ln x = \int_1^x \frac{1}{t} \, dt. \)

We can use this definition of \( \ln x \) to derive all the properties of \( \ln x \). This is an important example of how to derive properties from functions defined as integrals.

**Properties of \( \ln x \):**
1. \( \ln 1 = 0. \) (proof: obvious from definition)
2. \( \ln(ab) = \ln a + \ln b. \)
   **proof:** (uses change of variable and properties of integrals)
   \[ \ln(ab) = \int_1^{ab} \frac{1}{t} \, dt = \int_1^a \frac{1}{t} \, dt + \int_a^{ab} \frac{1}{t} \, dt \]
   For the second integral on the right let \( au = t \)
   \( \Rightarrow \quad a \, du = dt, \ t = a \leftrightarrow u = 1, \ t = ab \leftrightarrow u = b \)
   Thus \( \ln(ab) = \int_1^a \frac{1}{t} \, dt + \int_1^b \frac{1}{u} \, du = \ln a + \ln b \)
3. \( \ln x \) is increasing. (proof: derivative = \( \frac{1}{x} > 0 \))

(Won’t do the following in class.)
4. \( \ln(1/a) = -\ln a. \) (proof: \( 0 = \ln 1 = \ln(a \cdot \frac{1}{a}) = \ln a + \ln(1/a). \))
5. \( \ln x^n = n \ln x. \) (proof: \( \ln x^n = \ln(x \cdot x \cdot x \cdots x) \))
6. \( \ln x \to \infty \) as \( x \to \infty. \) (proof: \( \ln 2^n = n \ln 2 \to \infty \) and \( \ln x \) is increasing)

**More uses of the second fundamental theorem:**
**Example:** Sketch graph of \( F(x) = \int_0^x \frac{\sin^2 y}{1+y^2} \, dy. \)

Critical points:
\[
F'(x) = \frac{x^3 - 1}{x+1+2x^2},
\]
\[
F''(x) = 0 \quad \Rightarrow \quad x = 1.
\]
Special values:
\[
F'(0) = 0.
\]
Sign of \( F'(x) \)

\[
\begin{array}{c|c|c}
F' < 0 & 1 & F' > 0 \\
\hline
x & \quad \rightarrow \quad & x
\end{array}
\]

**Example:** 3D-8a) If \( \int_0^x f(t) \, dt = 2x(\sin x + 1) \) find \( f(\pi/2). \)
**answer:** \( f(x) = \) derivative of integral = \( \frac{d}{dx}2x(\sin x + 1) = 2(\sin x + 1) + 2x(\cos x) \)
\( \Rightarrow \ f(\pi/2) = 4. \)

**Example:** 3D-8b) If \( \int_0^{x^2} f(t) \, dt = 2x(\sin x + 1) \) find \( f(\pi/2). \)
**answer:** Chain rule: Let \( F(u) = \int_0^u f(t) \, dt. \)
So \( F'(u) = f(u) \) and \( \frac{d}{dx}F(x/2) = F'(x/2)\frac{1}{2} = f(x/2)\frac{1}{2}. \)
But, \( F(x/2) = 2x(\sin x + 1) \Rightarrow \frac{d}{dx}F(x/2) = 2(\sin x + 1) + 2x(\cos x) = \frac{1}{2}f(x/2). \)
So, \( \frac{d}{dx}F(x/2) = 2 + 2\pi(-1) = 2 - 2\pi \quad \Rightarrow \quad f(\pi/2) = 4 - 4\pi. \)

**Example:** 3D-11a) (change of variable):
Compute \( \int_1^e \sqrt{x^3} \, dx. \)
Substitute: \( u = \ln x \Rightarrow du = \frac{1}{x} \, dx, \quad x = 1 \leftrightarrow u = 0, \quad x = e \leftrightarrow u = 1. \)
\( \Rightarrow \) integral = \( \int_0^1 \sqrt{u} \, du = 2/3. \)

**Example:** 3D-11b) Compute \( \int_0^\pi \frac{\sin x}{(2+\cos x)^3} \, dx. \)
Substitute: \( u = \cos x \Rightarrow du = -\sin x \, dx, \quad x = 0 \leftrightarrow u = 1, \quad x = \pi \leftrightarrow u = -1. \)
\( \Rightarrow \) integral = \( \int_1^{-1} -\frac{1}{(2+u)^3} \, du = \int_1^{-1} \frac{1}{(2+u)^3} \, du = -\frac{1}{4}(2+u)^{-2}|_1^{-1} = 4/9. \)