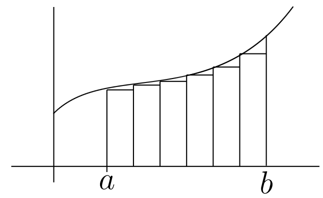
18.01A Topic 5: Second fundamental theorem, $\ln x$ as an integral.
Read: SN: PI, FT.

First Fundamental Theorem: $F' = f \;\Rightarrow\; \int_a^b f(x)\,dx = F(x)\big|_a^b$

Questions: 1. Why dx? 2. Given $f(x)$ does $F(x)$ always exist?

Answer to question 1.



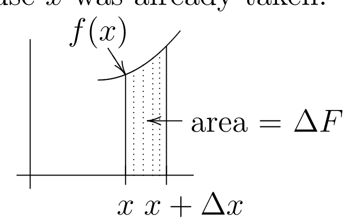
a) Riemann sum: Area $\approx \sum_1^n f(c_i)\Delta x \to \int_a^b f(x)\,dx$
In the limit the sum becomes an integral and the finite Δx becomes the infinitesimal dx.
b) The dx helps with change of variable.

Example: Compute $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$.
Let $x = \sin u$.
$\frac{dx}{du} = \cos u \;\Rightarrow\; dx = \cos u\,du, \quad x = 0 \;\Rightarrow\; u = 0, \quad x = 1 \;\Rightarrow\; u = \pi/2.$
Substituting: $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2 u}} \cos u\,du = \int_0^{\pi/2} \frac{\cos u}{\cos u}\,du = \int_0^{\pi/2} du = \pi/2.$

Answer to question 2. Yes →

> **Second Fundamental Theorem**:
> If f is continuous and $F(x) = \displaystyle\int_a^x f(u)\,du$ then $F'(x) = f(x)$.
> I.e. f *always* has an anit-derivative.

This is subtle: we have defined a NEW function $F(x)$ using the definite integral.
Note we needed a dummy variable u for integration because x was already taken.



proof:
$$\Delta\text{area} = \int_x^{x+\Delta x} f(x)\,dx = \int_0^{x+\Delta x} f(x)\,dx - \int_0^x f(x)\,dx = F(x+\Delta x) - F(x) = \Delta F.$$

But, also $\Delta\text{area} \approx f(x)\,\Delta x \;\Rightarrow\; \Delta F \approx f(x)\,\Delta x$ or $\dfrac{\Delta F}{\Delta x} \approx f(x)$.

As $\Delta x \to 0$ this becomes exact: $\frac{dF}{dx} = f(x)$. QED

More subtlety: For any continuous function there is an anti-derivative. We might not know it in closed form but we can always write it as a definite integral with a variable limit. This is useful since Riemann sums let us compute it as accurately as we wish.

Examples: (Not elementary functions BUT they are functions.)
$F(x) = \int_0^x e^{-t^2}\,dt$ statistics.
$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln t}\,dt$ number theory.
$\mathrm{Si}(x) = \int_0^x \sin(t^2)\,dt$ optics.

(continued)

Natural logarithm as a definite integral: $\ln x = \displaystyle\int_1^x \frac{1}{t}\,dt.$

We can use this definition of $\ln x$ to derive all the properties of $\ln x$. This is an important example of how to derive properties from functions defined as integrals.

Properties of $\ln x$:
1. $\ln 1 = 0$. (proof: obvious from definition)

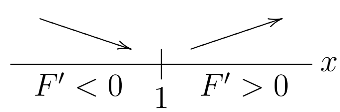
2. $\ln(ab) = \ln a + \ln b$.
proof: (uses change of variable and properties of integrals)
$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt$
For the second integral on the right let $au = t$
$\Rightarrow\ a\,du = dt,\ t = a \leftrightarrow u = 1,\ t = ab \leftrightarrow u = b$
Thus $\ln(ab) = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{au}\,a\,du = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{u}\,du = \ln a + \ln b$

3. $\ln x$ is increasing. (proof: derivative $= \frac{1}{x} > 0$)

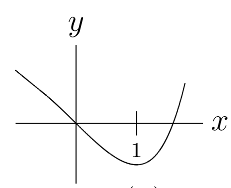
(Won't do the following in class.)
4. $\ln(1/a) = -\ln a$. (proof: $0 = \ln 1 = \ln(a \cdot \frac{1}{a}) = \ln a + \ln(1/a)$.)
5. $\ln x^n = n \ln x$. (proof: $\ln x^n = \ln(x \cdot x \cdot x \cdots x)\ldots$)
6. $\ln x \to \infty$ as $x \to \infty$. (proof: $\ln 2^n = n \ln 2 \to \infty$ and $\ln x$ is increasing)

More uses of the second fundamental theorem:
Example: Sketch graph of $F(x) = \int_0^x \frac{u^5-1}{1+u^2}\,du$.

Critical points:
$F'(x) = \frac{x^5-1}{1+x^2}$.
$F'(x) = 0 \Rightarrow x = 1$.
Special values:
$F(0) = 0$.



Sign of $F'(x)$



$y = F(x)$

Example: 3D-8a) If $\int_0^x f(t)\,dt = 2x(\sin x + 1)$ find $f(\pi/2)$.
<u>**answer:**</u> $f(x) =$ derivative of integral $= \frac{d}{dx} 2x(\sin x + 1) = 2(\sin x + 1) + 2x(\cos x)$
$\Rightarrow\ f(\pi/2) = 4$.

Example: 3D-8b) If $\int_0^{x/2} f(t)\,dt = 2x(\sin x + 1)$ find $f(\pi/2)$.
<u>**answer:**</u> Chain rule: Let $F(u) = \int_0^u f(t)\,dt$.
So $F'(u) = f(u)$ and $\frac{d}{dx} F(x/2) = F'(x/2)\frac{1}{2} = f(x/2)\frac{1}{2}$.
But, $F(x/2) = 2x(\sin x + 1) \Rightarrow \frac{d}{dx} F(x/2) = 2(\sin x + 1) + 2x(\cos x) = \frac{1}{2} f(x/2)$.
So, let $x = \pi \Rightarrow \frac{1}{2} f(\pi/2) = 2 + 2\pi(-1) = 2 - 2\pi \Rightarrow f(\pi/2) = 4 - 4\pi$.

Example: 3D-11a) (**change of variable**):
Compute $\int_1^{e} \frac{\sqrt{\ln x}}{x}\,dx$.
Substitute: $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$, $x = 1 \leftrightarrow u = 0$, $x = e \leftrightarrow u = 1$.
$\Rightarrow$ integral $= \int_0^1 \sqrt{u}\,du = 2/3$.

Example: 3D-11b) Compute $\int_0^{\pi} \frac{\sin x}{(2+\cos x)^3}\,dx$.
Substitute: $u = \cos x \Rightarrow du = -\sin x\,dx$, $x = 0 \leftrightarrow u = 1$, $x = \pi \leftrightarrow u = -1$.
$\Rightarrow$ integral $= \int_1^{-1} -\frac{1}{(2+u)^3}\,du = \int_{-1}^1 \frac{1}{(2+u)^3}\,du = -\frac{1}{2}(2+u)^{-2}\Big|_{-1}^1 = 4/9$.