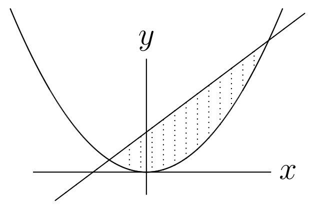
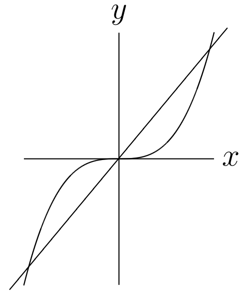
18.01A Topic 6: Geometric applications: volumes, area, arclength.
Read: TB: 7.1-7.6 (7.4 to top of page 233, 7.5 lightly).

Idea:
1. Divide into little (infinitesimal) pieces, each of which is easy to compute.
2. Sum them up.

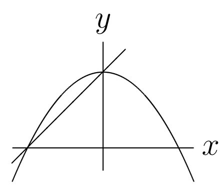
Area between curves
Examples: (won't do all in class)
1. Find the area between $y = x + 2$ and $y = x^2$.
answer: Find points of intersection:
$x + 2 = x^2 \Rightarrow x = -1,\ 2.$
On top: $x + 2$ above x^2.
Area $= \int_{-1}^{2} (x + 2) - x^2\,dx = 9/2$.





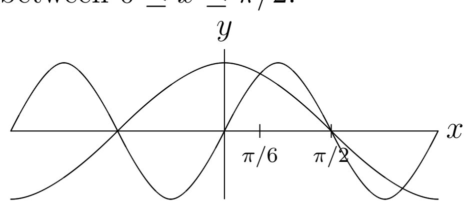
2. Find the area between curves $y = x^3$ and $y = 4x$.
Intersect at $x = -2, 0, 2$.
Two pieces: $\int_{-2}^{0} x^3 - 4x\,dx + \int_{0}^{2} 4x - x^3\,dx = 8$.

3. Find the area between $y = 1-x^2$ and $y = x+1$.
Intersection: $1 - x^2 = x + 1 \Rightarrow x = 0,\ -1.$
On top: $1 - x^2$ is above $x + 1$.
Area $= \int_{-1}^{0} (1 - x^2) - (x + 1)\,dx = 1/6$.



4. Find the area bounded by $y = \cos x$, $y = \sin 2x$, between $0 \le x \le \pi/2$.

Intersection: $\cos x = \sin 2x = 2 \sin x \cos x$
$\Rightarrow \cos x = 0$ or $2 \sin x = 1 \Rightarrow x = \pi/2, \pi/6$.
In $[0, \pi/6]$ $\cos x$ is on top.
In $[\pi/6, \pi/2]$ $\sin 2x$ is on top.

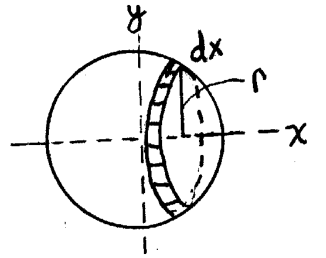


$$\text{Area} = \int_0^{\pi/6} \cos x - \sin 2x + \int_{\pi/6}^{\pi/2} \sin 2x - \cos x$$
$$= \sin x + \tfrac{1}{2}\cos 2x\Big|_0^{\pi/6} + -\tfrac{1}{2}\cos 2x - \sin x\Big|_{\pi/6}^{\pi/2}$$
$$= (\tfrac{1}{2} - 0 + \tfrac{1}{4} - \tfrac{1}{2}) + (\tfrac{1}{2} - 1 + \tfrac{1}{4} + \tfrac{1}{2}) = \tfrac{1}{2}.$$

(continued)

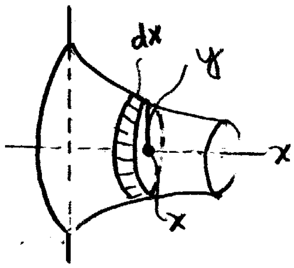
Volume by Slices:

Example 1: Find the volume of the sphere of radius R.



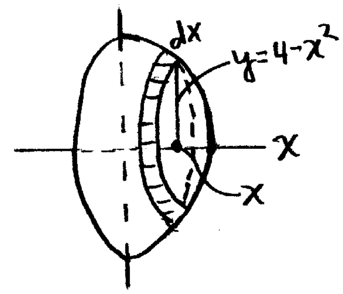
Volume of slice: $dV = \pi r^2\,dx = \pi(R^2 - x^2)\,dx$.
Volume = 'sum' of slices:
$V = \int_{-R}^{R} \pi(R^2 - x^2)\,dx$
$= \pi(R^2x - x^3/3)\big|_{-R}^{R} = \frac{4}{3}\pi R^3$.

Volume of revolution:



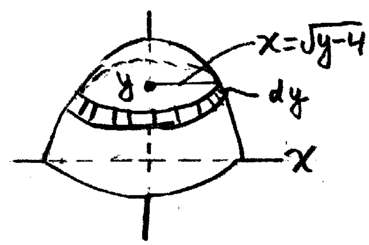
Revolve graph of $y = f(x)$ around x-axis.
Compute volume by vertical slices.
Volume of slice: $dV = \pi y^2\,dx = \pi f(x)^2\,dx$
Volume = 'sum' of slices:
$V = \int_a^b \pi f(x)^2\,dx$

Example 2: . Find the volume of revolution of the curve $y = 4 - x^2$ between 0 and 2 revolved around the x-axis.



Volume of slice: $dV = \pi y^2\,dx = \pi(4 - x^2)^2\,dx$.
Volume = 'sum' of slices:
$V = \int_0^2 \pi 16 - 8x^2 + x^4\,dx$
$= \pi(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5)\big|_0^2$
$= \pi(32 - 64/3 + 32/5)$.

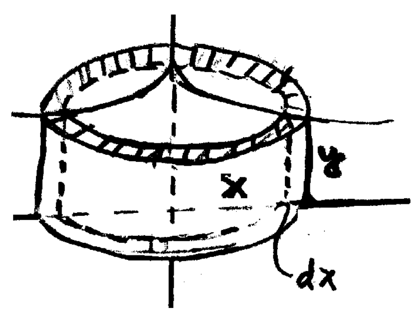
Example 3: . Same curve around y-axis.



Volume of slice: $dV = \pi x^2\,dy = \pi(4 - y)\,dy$.
Volume = 'sum' of slices:
$V = \int_0^4 \pi(4 - y)\,dy = \pi(4y - y^2/2)\big|_0^4 = 8\pi$.

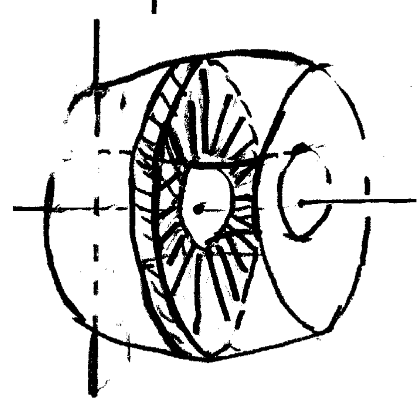
Volume by Shells

Example 4: Same curve ($y = 4 - x^2$) around y-axis.



Vertical rectangle in first graph sweeps out cylindrical shell.
thickness = dx, height = y, radius = x.
$\Rightarrow$ Volume of shell = $dV = 2\pi xy\,dx$.
$\Rightarrow$ Volume = 'sum' of shells:
$V = \int_0^2 2\pi x(4 - x^2)\,dx$
$= 2\pi(2x^2 - x^4/4)\big|_0^2 = 8\pi$.
(Same as by slices!)

Washers

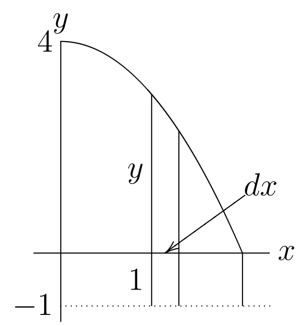


Revolve area between curves around the x-axis.
Just like slices with inside and outside radii.
Volume of slice = $dV = \pi(y_1^2 - y_2^2)\,dx$.
Volume = 'sum' of slices = $\int_a^b dV$.

(continued)

The same ideas work for volumes of revolution around other lines.

Example 5: Find the volume of revolution of the curve $y = 4 - x^2$ between $x = 0$ and $x = 2$ around the line $y = -1$.



answer: This is the same as example 2 above, except instead of rotating around the line $y = 0$ (the x-axis) we rotate around $y = -1$.

The volume of a thin disk of rotation is

$dV = \pi r^2 dx = \pi(y+1)^2 dx = \pi(5 - x^2)^2\, dx$.

Volume = 'sum' of slices:

$$\begin{aligned} V &= \int_0^2 \pi(25 - 10x^2 + x^4\, dx \\ &= \pi(25x - \tfrac{10}{3}x^3 + \tfrac{1}{5}x^5|_0^2 \\ &= \pi(50 - \tfrac{80}{3} + \tfrac{32}{5}). \end{aligned}$$

Arclength

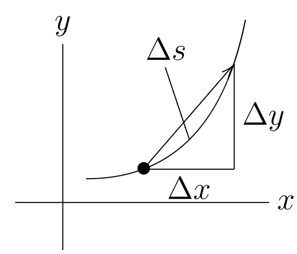
Δs is the length along the curve.

It's approximated by the secant line.

I.e. $\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$

In the limit: $ds = \sqrt{(dx)^2 + (dy)^2}$

This is the basic formula. It can be manipulated.



E.g. $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ or $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$.

$$\boxed{\text{Arclength} = L = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

Example: Find the arclength of the curve $y^2 = x^3$ between $(0,0)$ and $(4,8)$.

answer: $y = x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{1/2} \Rightarrow \frac{ds}{dx} = \sqrt{1 + \frac{9}{4}x}$.

$\Rightarrow$ Arclength $= L = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{8}{27}(1 + \frac{9}{4}x)^{3/2}\big|_0^4 = \frac{8}{27}(10^{3/2} - 1)$.

Example: Find the arclength of $y = \sin x$ for x in $[0, \pi]$.

answer: $\frac{dy}{dx} = \cos x \Rightarrow \frac{ds}{dx} = \sqrt{1 + \cos^2 x}$.

$\Rightarrow$ Arclength $= L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx$.

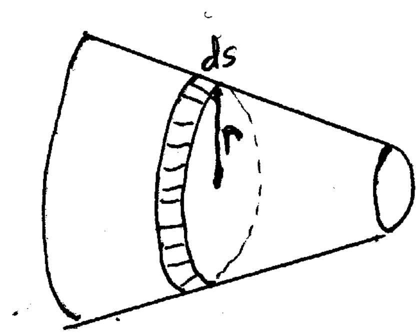
(Not possible to evaluate in terms of elementary functions –called an elliptic integral.)

(continued)

Surface area of revolution:
Trickier than volume.
Main idea: rotate line to get frustrum of cone: read the book §7.6.

For a curve $y = f(x)$ rotated around the x-axis the differential of area is
$dA = 2\pi y\, ds = 2\pi y\sqrt{1 + (y')^2}\, dx$.

In general, for a element ds rotated in a circle of radius r we get $dA = 2\pi r\, ds$.



Example: Find the area of the surface of revolution of $y = 1/x$ between 1 and b revolved around the x-axis.
answer: Find dA:
$y' = -1/x^2$.
$ds = \sqrt{1 + (y')^2}\, dx = \sqrt{1 + (-1/x^2)^2}\, dx$.
$dA = 2\pi y\, ds = 2\pi \frac{1}{x}\sqrt{1 + (-1/x^2)^2}\, dx$.
$\Rightarrow$ Surface area $A = \int_1^b 2\pi \frac{1}{x}\sqrt{1 + 1/x^4}\, dx$.
Hard to compute but we can analyze:
$A > \int_1^b 2\pi \frac{1}{x}\, dx = 2\pi \ln b$.
$\Rightarrow$ $A \to \infty$ as $b \to \infty$.

Find the volume of revolution of the same curve.
Volume of slice $= dV = \pi y^2\, dx = \pi \frac{1}{x^2}\, dx$.
$\Rightarrow$ $V = \int_1^b \frac{\pi}{x^2}\, dx = -\frac{\pi}{x}\Big|_1^b = \pi(1 - \frac{1}{b})$.
$\Rightarrow$ $V \to \pi$ as $b \to \infty$.

Finite volume and infinite surface area!
What happens if you fill the volume of revolution with paint?