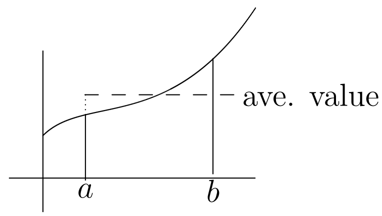
18.01A Topic 7: More applications: work, average value.
Read: TB: 7.7 to middle p. 247, SN: AV.

Average Value:
Definition: Average value of f over $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Idea: Ave $\approx \sum_1^n f(x_i)\frac{1}{n} = \frac{1}{b-a}\sum f(x_i)\frac{b-a}{n} = \frac{1}{b-a}\sum f(x_i)\Delta x \to \frac{1}{b-a}\int_a^b f(x)\,dx$.

Examples:
1. Ave. val. of x on $[0, 3] = \frac{1}{3}\int_0^3 x\,dx = 3/2$.
Ave. val. of x on $[5, 10] = \frac{1}{5}\int_5^{10} x\,dx = 15/2$.
2. Ave. val. of $\sin x$ on $[0, \pi] = \frac{1}{\pi}\int_0^\pi \sin x\,dx = 2/\pi$.

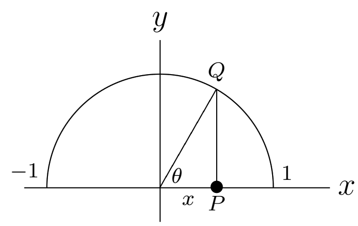


More idea:
Average value = 'constant with the same integral'.
What you average over is important. The following is taken from the notes §AV.
Example: (This will be done at the end of class if there's time.)
A point is chosen at random on the x-axis between -1 and 1; call it P. What is the average length of the vertical line from P to the unit circle?

<u>**answer:**</u> Random on the x-axis means averaging over x.

If P is at x the line has length $\sqrt{1-x^2}$.

$$\Rightarrow \text{ ave. length} = \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx$$
$$= \frac{1}{2}\text{ area of semicircle} = \frac{\pi}{4}.$$



Now take Q randomly on the circumference of the unit semicircle.
What is the average length of the vertical line from Q to the x-axis?

<u>**answer:**</u> Now we must average over θ.

For fixed θ the line has length $= \sin\theta \;\Rightarrow$ ave. length $= \dfrac{1}{\pi}\displaystyle\int_0^\pi \sin\theta\,d\theta = \dfrac{2}{\pi}$.

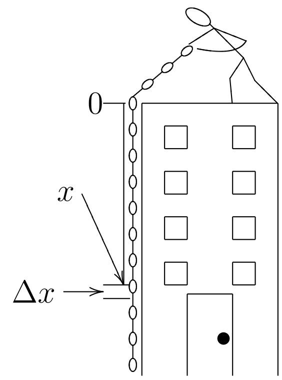
The two averages are different!

Work:
For constant force: work = force time distance.
Units = energy: ft-lb, joule = newton-meter, erg = dyne-cm.
Hooke's law: For a spring $F = -kx$ where F = force needed to compress or stretch a spring x units from equilibrium. k is the spring constant (the bigger k the stiffer the spring).
Gravitation: $F = \frac{GmM}{x^2}$ where F is the gravitational attraction of two masses m and M a distance x apart. G is the universal constant of gravitation.

(continued)

Examples:

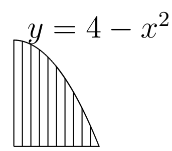
1. A 100 ft long chain weighing 5 lb/ft hangs off a building. How much work is expended to haul it up to the roof?
Mass of little segment $= \Delta m = 5\,\Delta x$.
$x =$ distance from roof to little segment.
Work to haul a little segment of chain to the roof:
$\Delta W = x\,\Delta m = 5\,\Delta x$
Total work $W \approx \sum \Delta W = \sum 5x\,\Delta x$.
Letting Δx go to 0 this becomes an integral:
$W = \int_0^{100} 5x\,dx = 25{,}000\,$lb-feet.



Alternate method: ΔW = work to move all of chain still hanging a little way.
$x =$ length of chain already hauled in.
Mass of remaining chain $= 5(100 - x)$.
$\Delta W = 5(100 - x)\,dx$.
$W = \int_0^{100} 5(100 - x)\,dx = 25{,}000\,$lb-feet.

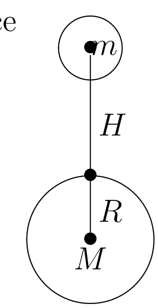
2. A spring has natural length 10 in., 12 lb. of force stretches it 1/2 in.
Find the work done in stretching it from 10 to 18 in.
__**answer:**__ First find k: 12 lb $= k \cdot \frac{1}{2}$in. $\Rightarrow$ $k = 24$ lb/in.
Work done stretching it from x to $x + \Delta x$ is $\Delta W \approx kx\Delta x$.
Total work $W = \sum \Delta W \approx \sum kx\Delta x$.
Letting $\Delta x \to 0$:
$W = \int_{10}^{18} 24x\,dx = 12x^2\big|_{10}^{18} = 2688$ lb-in $= 224$ lb-ft.
(Note: always be careful with units.)

Example: (Average Value)
Given a harp as shown, what is the average length of a string?



Ave $= \sum f(c_i)\frac{1}{n} = \frac{1}{2}\sum f(c_i)\frac{2}{n} \approx \frac{1}{2}\int_0^2 f(x)\,dx = 8/3$.
Inverse kind of problem: use integral to approximate finite sum.

Example: (Gravitation)
How much work is done in moving a mass m from the surface of the planet to a height H?



Work done going from h to $h + \Delta h = \Delta W \approx \frac{GmM}{R+h)^2}\,\Delta h$.
Total work $W = \sum \Delta W \approx \sum \frac{GmM}{(R+h)^2}\,\Delta h$.
Limit as $\Delta h \to 0$:
$W = \int_0^H \frac{GmM}{R+h)^2}\,dh = -\frac{GmM}{R+h}\Big|_0^H = GmM(\frac{1}{R} - \frac{1}{R+H})$
A high point of Western civilization!