18.01A Topic 7: More applications: work, average value.
Read: TB: 7.7 to middle p. 247, SN: AV.

Average Value:
Definition: Average value of $f$ over $[a, b]$ is \( \frac{1}{b-a} \int_a^b f(x) \, dx \).

Idea: $\text{Ave} \approx \sum \frac{1}{n} f(x_i) \Delta x \to \frac{1}{b-a} \int_a^b f(x) \, dx$.

Examples:
1. Ave. val. of $x$ on $[0, 3] = \frac{1}{3} \int_0^3 x \, dx = 3/2$.
2. Ave. val. of $x$ on $[5, 10] = \frac{1}{5} \int_5^{10} x \, dx = 15/2$.

More idea:
Average value = 'constant with the same integral'.

What you average over is important. The following is taken from the notes §AV.

**Example:** (This will be done at the end of class if there's time.)
A point is chosen at random on the $x$-axis between -1 and 1; call it $P$. What is the average length of the vertical line from $P$ to the unit circle?

**Answer:** Random on the $x$-axis means averaging over $x$.

If $P$ is at $x$ the line has length $\sqrt{1-x^2}$.

\[
\Rightarrow \text{ave. length} = \frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} \, dx
\]

\[= \frac{1}{2} \text{ area of semicircle} = \frac{\pi}{4}.\]

Now take $Q$ randomly on the circumference of the unit semicircle.
What is the average length of the vertical line from $Q$ to the $x$-axis?

**Answer:** Now we must average over $\theta$.

For fixed $\theta$ the line has length $= \sin \theta \Rightarrow \text{ave. length} = \frac{1}{\pi} \int_0^\pi \sin \theta \, d\theta = \frac{2}{\pi}$.

The two averages are different!

Work:
For constant force: work = force time distance.
Units = energy: ft-lb, joule = newton-meter, erg = dyne-cm.

**Hooke’s law:** For a spring $F = -kx$ where $F$ = force needed to compress or stretch a spring $x$ units from equilibrium. $k$ is the spring constant (the bigger $k$ the stiffer the spring).

**Gravitation:** $F = \frac{GmM}{x^2}$ where $F$ is the gravitational attraction of two masses $m$ and $M$ a distance $x$ apart. $G$ is the universal constant of gravitation.

(continued)
Examples:
1. A 100 ft long chain weighing 5 lb/ft hangs off a building. How much work is expended to haul it up to the roof?
Mass of little segment = \( \Delta m = 5 \Delta x \).

Work to haul a little segment of chain to the roof:
\[
\Delta W = x \Delta m = 5 \Delta x
\]
Total work \( W \approx \sum \Delta W = \sum 5x \Delta x \).
Letting \( \Delta x \) go to 0 this becomes an integral:
\[
W = \int_0^{100} 5x \, dx = 25,000 \text{ lb-feet}.
\]
Alternate method: \( \Delta W = \) work to move all of chain still hanging a little way.
\[
x = \text{length of chain already hauled in}.
\]
Mass of remaining chain = \( 5(100 - x) \).
\[
\Delta W = 5(100 - x) \, dx.
\]
\[
W = \int_0^{100} 5(100 - x) \, dx = 25,000 \text{ lb-feet}.
\]

2. A spring has natural length 10 in., 12 lb. of force stretches it 1/2 in.
Find the work done in stretching it from 10 to 18 in.
answer: First find \( k \): 12 lb = \( k \cdot \frac{1}{2} \text{in.} \Rightarrow k = 24 \text{ lb/in.} \)
Work done stretching it from \( x \) to \( x + \Delta x \) is \( \Delta W \approx kx \Delta x \).
Total work \( W = \sum \Delta W \approx \sum kx \Delta x \).
Letting \( \Delta x \to 0 \):
\[
W = \int_{10}^{18} 24x \, dx = 12x^2 \bigg|_{10}^{18} = 2688 \text{ lb-in} = 224 \text{ lb-ft}.
\]
(Note: always be careful with units.)

Example: (Average Value)
Given a harp as shown, what is the average length of a string?
\[
\text{Ave} = \sum f(c_i) \frac{1}{n} = \frac{1}{2} \sum f(c_i) \frac{2}{n} \approx \frac{1}{2} \int_0^2 f(x) \, dx = \frac{8}{3}.
\]
Inverse kind of problem: use integral to approximate finite sum.

Example: (Gravitation)
How much work is done in moving a mass \( m \) from the surface of the planet to a height \( H \)?
Work done going from \( h \) to \( h + \Delta h = \Delta W \approx \frac{GmM}{(R+h)^2} \Delta h \).
Total work \( W = \sum \Delta W \approx \sum \frac{GmM}{(R+h)^2} \Delta h \).
Limit as \( \Delta h \to 0 \):
\[
W = \int_0^H \frac{GmM}{(R+h)^2} \, dh = -\frac{GmM}{R} \bigg|_0^H = GmM \left( \frac{1}{R} - \frac{1}{R+H} \right)
\]
A high point of Western civilization!