18.01A Topic 8: Integration: substitution, trigonometric integrals, completing the square.
Read: TB: 10.2, 10.3, 10.4.

Integration techniques Only practice will make perfect. These techniques are important, but *not* the intellectual heart of the class.

1. Inspection: $\int x^2\,dx = \frac{x^3}{3} + C$.

2. Guess/memorize: $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$. **Memorize this!**

3. Direct substitution: $u = g(x) \Rightarrow du = g'(x)\,dx$.

Examples:
a. Compute $\int \sin^4 x \cos x\,dx$.
answer: Let $u = \sin x \Rightarrow du = \cos x\,dx$.
Substitute for all pieces in integral: $\Rightarrow \int u^4\,du = u^5/5 + C$.
Back substition: integral $= \frac{\sin^5 x}{5} + C$.

b. Compute $\int \sin^5 x\,dx$.
answer: $\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx$.
Let $u = \cos x \Rightarrow du = -\sin x\,dx$.
Substitute: integral $= -\int (1-u^2)^2\,du$ (Easy to compute and back substitute)

Trig formulas you have to know:
1. $\sin^2 x + \cos^2 x = 1$.

2. Double angle: $\sin 2x = 2\sin x\cos x \quad \cos 2x = \cos^2 x - \sin^2 x$.

3. Half angle: $\cos^2 x = \frac{1}{2}(1+\cos 2x) \quad \sin^2 x = \frac{1}{2}(1-\cos 2x)$.

4. $1 + \tan^2 x = \sec^2 x \quad \sec^2 x - 1 = \tan^2 x$.

5. $\frac{d}{dx}\sin x = \cos x \quad \frac{d}{dx}\cos x = \sin x$.
$\frac{d}{dx}\tan x = \sec^2 x \quad \frac{d}{dx}\sec x = \sec x\tan x$.

Examples:
1. Compute $\int \sin^4 x\,dx$.
answer: $\sin^4 x = (\frac{1-\cos 2x}{2})^2 = \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x = \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}(1+\cos 4x)$.
This last expression is easy to integrate.

2. Compute $\int \tan^2 x \sec^2 x\,dx$.
answer: Notice $\sec^2 x$ is the derivative of $\tan x$.
$\Rightarrow$ substitute $u = \tan x,\ du = \sec^2 x\,dx$.
$\Rightarrow$ integral $= \int u^2\,du = u^3/3 + C$.
Back substitute: integral $= \tan^3 x/3 + C$.

(continued)

3. Compute $\int \tan^2 x \sec^4 x\, dx$.
answer: Notice $\sec^4 x = (1+\tan^2 x)\sec^2 x = (1+\tan^2 x)\, d\tan x$.
Substitute $u = \tan x$
$\Rightarrow$ integral $= \int u^2(1+u^2)\, du$. (Easy to integrate and back substitute.)

Inverse trig substitution Examples:
1. Compute $\int \frac{1}{\sqrt{1-x^2}}$.
answer: Substitute $x = \sin\theta \;\Rightarrow\; dx = \cos\theta\, d\theta \;\Rightarrow\;$ integral $= \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$.
(Notice how easy the substitution is.)
Trig identities $\Rightarrow$ integral $= \int \frac{1}{\cos\theta}\cos\theta\, d\theta = \int d\theta = \theta + C$.
Back substitution (Use $\theta = \sin^{-1} x$) $\Rightarrow$ integral $= \sin^{-1} x + C$.
Check answer by differentiation.

2. Compute $\int \frac{1}{\sqrt{a^2-x^2}}$.
answer: Substitute $x = a\sin\theta \;\Rightarrow\; dx = a\cos\theta\, d\theta$.
Now similar to example 1.

3. Compute $\int \frac{1}{\sqrt{a^2+x^2}}$.
answer: Substitute $x = a\tan\theta \;\Rightarrow\; dx = a\sec^2\theta\, d\theta$
$\Rightarrow$ integral $= \int \frac{1}{\sqrt{a^2+a^2\tan^2\theta}} a\sec^2\theta\, d\theta$.
Algebra + trig identities $\Rightarrow$
integral $= \int \frac{1}{\sec\theta}\sec^2\theta\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$
Back substitution (Use $\sec\theta = \sqrt{1+(x/a)^2})$) $\Rightarrow$
integral $= \ln|\sqrt{1+(x/a)^2} + x/a| + C = \ln|\sqrt{a^2+x^2}+x| - \ln a + C$
$= \ln|\sqrt{a^2+x^2}+x| + C$.
(In the last equality we replaced the constant $\ln a + C$ by C.)
Check answer by differentiation.

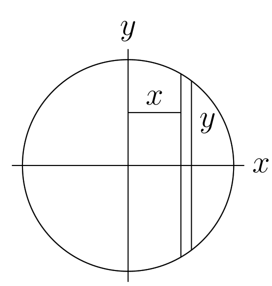
Example: Moment of inertia of uniform disk of radius a around a diameter.
Moment of inertia of point mass about an axis is $I = md^2$,
where m is the mass and d is the distance from the axis.

Choose the diameter to be along the y-axis.

Let total mass of disk $= M$

(so uniform density $\delta = M/(\pi a^2)$).

Usual slice and sum technique:



Vertical strip is (approximately) a distance x from axis.

Area of strip $= dA = 2y\, dx$.

Mass of strip $= dm = \delta\, dA = \delta 2y\, dx$.

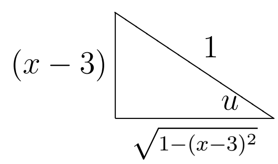
Moment of inertia of strip $= dI = x^2\, dm = x^2\delta 2y\, dx$.

Total moment of inertia $= I = \int 2\delta x^2 y\, dx$.

(continued)

Always use symmetry –it suffices to compute for half disk (and multiply by 2).
Adding limits and $y = \sqrt{a^2 - x^2}$: $\quad I = 2\int_0^a 2\delta x^2\sqrt{a^2 - x^2}\,dx$.
Substitute $x = a\sin\theta$, $\quad dx = a\cos\theta\,d\theta$.
$\Rightarrow\ x = 0 \leftrightarrow \theta = 0,\ x = a \leftrightarrow \theta = \pi/2$.
$\Rightarrow\ I = 4\delta\int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^2 - a^2\sin^2\theta}\,a\cos\theta\,d\theta = 4\delta\int_0^{\pi/2} a^4\sin^2\theta\cos^2\theta\,d\theta$.
Trig identities: $\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2 2\theta = \frac{1}{8}(1 - \cos 4\theta)$
$\Rightarrow\ I = \frac{1}{2}a^4\delta\int_0^{\pi/2} 1 - \cos 4\theta\,d\theta = \frac{1}{2}a^4\delta(\theta - \sin(4\theta)/4)\Big|_0^{\pi/2} = a^4\delta\pi/4 = Ma^2/4$.

Completing the square:
(This is how you derive the quadratic formula)
Example: Compute $\int \frac{1}{x^2+2x+5}\,dx$.
Complete the square: $\quad x^2 + 2x + 5 = x^2 + 2x + 1 + 5 - 1 = (x+1)^2 + 4$.
$\Rightarrow\ \int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx$
Substitute $x + 1 = 2\tan u,\ dx = 2\sec^2 u\,du$
$\Rightarrow$ integral $= \int \frac{2\sec^2 u}{4\sec^2 u}\,du = \frac{1}{2}u + C = \frac{1}{2}\tan^{-1}(\frac{x+1}{2}) + C$
Example: 5D-11. Compute $\int x\sqrt{-8 + 6x - x^2}\,dx$.
<u>**answer:**</u> Complete the square inside the square root:
$-(x^2 - 6x + 8) = -(x^2 - 6x + 9 - 9 + 8) = -(x-3)^2 + 1$.
$\Rightarrow$ integral $= \int x\sqrt{1 - (x-3)^2}\,dx$ (Substitue $\sin u = x - 3,\ \cos u\,du = dx$.)
$= \int(\sin u + 3)\cos^2 u\,du$.
Two pieces:
i) $\int \sin u\cos^2 u\,du = -\frac{1}{3}\cos^3 u$.
ii $\int 3\cos^2 u\,du = \frac{3}{2}\int 1 + \cos 2u\,du = \frac{3}{2}(u + \sin(2u)/2)$.



$\Rightarrow$ integral $= -\frac{1}{3}\cos^3 u - \frac{3}{4}\sin 2u - \frac{3}{2}u + C$
$= -\frac{1}{3}(-8 + 6x - x^2)^{3/2} + \frac{3}{2}(x+3)(-8 + 6c - x^2)^{1/2} + \frac{3}{2}\sin^{-1}(x-3) + C$.