18.01A Topic 9: Integration: partial fractions.
Read: TB: 10.6, SN: F

Example: $\dfrac{4}{x-3} - \dfrac{1}{x-1} = \dfrac{3x-1}{x^2-4x+3}$.

Question: How do we go backwards?

$$\frac{3x-1}{x^2-4x+3} = \frac{3x-1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}.$$

$\Rightarrow\ 3x-1 = A(x-1) + B(x-3)$.

$x = 1$: $\Rightarrow\ B = -1$.

$x = 3$: $\Rightarrow\ A = 4$.

Coverup method: Doing the above without writing. I don't like it because I make mistakes. Read notes §F.

Example: Compute $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$.

answer: Partial fractions: $\frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$.

Cross multiply: $1 = A(x-2) + B(x-1)$.

$x = 1\ \Rightarrow\ A = -1$.

$x = 2\ \Rightarrow\ B = 1$.

$\Rightarrow$ integral $= \int -\frac{1}{x-1} + \frac{1}{x-2}\,dx = -\ln(x-1) + \ln(x-2) + C$.

Getting fancier:

Repeated linear factors:

Example: $\frac{x^2-2x+2}{(x+2)^2(x-2)} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x-2}$ –One term for each power.

Multiply out and solve:

$x^2 - 2x + 2 = A(x-2) + B(x+2)(x-2) + C(x+2)^2$

$x = 2\ \Rightarrow\ C = 1/8$

$x = -2\ \Rightarrow\ A = -5/2$

Out of obvious substitutions. Many ways to proceed.

One way: $x = 0\ \Rightarrow\ 2 = -2A - 4B + 4C\ \Rightarrow\ B = 7/8$.

(continued)

Quadratic factor:
Example: $\frac{x^2+3}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$
Multiply out and solve:
$x^2 + 3 = A(x^2 + 1) + (Bx + C)(x - 1)$
$x = 1 \Rightarrow A = 2$
Out of obvious substitions. Many ways to proceed.
One way: $x = 0 \Rightarrow C = -1$.
Equate coefficient of x^2 on each side $\Rightarrow 1 = A + B \Rightarrow B = -1$.
Second way: Equate coefficients: $x^2 + 3 = (A + B)x^2 + (C - B)x + (A - C)$.
$\Rightarrow 1 = A + B,\ 0 = C - B,\ 3 = A - C\ \ldots$
(This last method is really what's going on no matter how you solve it.)

Integration example:

$$\begin{aligned}\int \frac{x^2+3}{(x^2+1)(x-1)}\,dx &= \int \frac{2}{x-1}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\ &= 2\ln|x-1| - \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x + C.\end{aligned}$$

(Should memorize $\int \frac{1}{x^2+1} = \tan^{-1} x + C$.)

Repeated Quadratic Factor:
Example: $\frac{2x+1}{(x^2+1)^2(x-1)} = \frac{Ax+B}{(x^2+1)^2} + \frac{Cx+D}{x^2+1} + \frac{E}{x-1}$
Multiply out: $2x + 1 = (Ax + B)(x - 1) + (Cx + D)(x^2 + 1)(x - 1) + E(x^2 + 1)^2$
$x = 1 \Rightarrow E = 3/4$
Equate coefficients: (painful calculation)
$x^4:\ 0 = C + E \Rightarrow C = -3/4$
$x^3:\ 0 = D - C \Rightarrow D = -3/4$
$x^2:\ 0 = A + C - D + 2E \Rightarrow A = -3/2$
$x:\ 2 = -A + B - C + D \Rightarrow B = 1/2$
$1:\ 1 = -B - D + E$ (this checks out)

Example: (more integration) Compute $\int \frac{x}{(x^2+1)^2}\,dx$.
Substitute $u = x^2 + 1,\ du = 2x\,dx$.
$\Rightarrow$ integral $= \int \frac{1}{2}\frac{1}{u^2}\,du = -\frac{1}{2}u^{-1} + C = -\frac{1}{2}(x^2+1)^{-1} + C$.

Long division
For partial fractions must have degree of numerator < degree of denominator.

Example: Decompose $\dfrac{x^3 + 2x + 1}{x^2 + x - 2} = \dfrac{x^3 + 2x + 1}{(x + 2)(x - 1)}$ using partial fractions.

First use long division

$$\begin{array}{r|rrrr} & x & -1 & & \\ \hline x^2+x-2 & x^3 & + & 2x & +1 \\ & x^3 & +x^2 & -2x & \\ \hline & & -x^2 & +4x & +1 \\ & & -x^2 & -x & +2 \\ \hline & & & 5x & -1 \end{array} \quad \Rightarrow \quad \frac{x^3 + 2x + 1}{x^2 + x - 2} = x - 1 + \frac{5x - 1}{x^2 + x - 2}.$$

Now proceed as always.