18.330 Problem Set 6

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Problem 1 *(Source code for this problem attached as* problem1.m, problem1.py, *and* mesh.py*)*

The construction of the mesh is as follows. At each mesh point (i, j), the differential equation can be approximated by

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = (\frac{1}{1+N})^2,$$

and we can construct an N^2xN^2 matrix M_N where $M_{(a-1)N+b,(c-1)N+d}$ represents the coefficient of $u_{c,d}$ in the equation centered at $u_{a,b}$. Let b_N be an N^2-tall column vector with each entry $\frac{1}{(N+1)^2}$. For example,

$$M_2 = \begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{1}{9} \\ \frac{1}{9} \\ \frac{1}{9} \end{pmatrix},$$

We solve $M_N x = b_N$ using software for $N = 3, 7, 15, 31, 63$ and extrapolate $u_{(\frac{N+1}{2}),(\frac{N+1}{2})}$ four times to get, to 10 digits, **-0.07367135327**. (Both MATLAB and SciPy give the same digits.)

Problem 2

We first define a few terms. Let

$$M_n = \begin{pmatrix} x_{n-1} - \frac{c_{n-1}}{c_n} & -\frac{c_{n-1}}{c_n} & \dots & \dots & -\frac{c_0}{c_n} \\ 1 & x_{n-2} & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & x_0 \end{pmatrix},$$

and $f_n(x) = det(xI_n - M_n)$. We will prove that $f_n(x) = \frac{1}{c_n}P_n(x)$, where

$$P_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \dots + c_n(x - x_0)\dots(x - x_{n-1}),$$

and that therefore the eigenvalues of M_n are the roots of $P_n(x)$ (requirement *(a)*). Note also that M_n is in upper Hessenberg form (requirement *(b)*) and that each entry of the matrix can be computed by simple arithmetic on the $\{c_i\}$ and $\{x_i\}$ (requirement *(c)*).

Conjecture. M_n is the companion matrix to P_n. That is,

$$f_n(x) = \frac{1}{c_n}P_n(x).$$

Proof. By induction. Our anchor is at $n = 2$.

$$f_2(x) = det\begin{pmatrix} x - x_1 + \frac{c_1}{c_2} & \frac{c_0}{c_2} \\ -1 & x - x_0 \end{pmatrix} = (x - x_1 + \frac{c_1}{c_2})(x - x_0) - (\frac{c_0}{c_2})(-1)$$

$$= (x - x_1)(x - x_0) + \frac{c_1}{c_2}(x - x_0) + \frac{c_0}{c_2} = \frac{1}{c_2}P_2(x)$$

Now we must show that $f_{n-1}(x) = \frac{1}{c_{n-1}}P_{n-1}(x) \Rightarrow f_n(x) = \frac{1}{c_n}P_n(x)$. Consider f_n:

$$f_n = det\begin{pmatrix} x - x_{n-1} + \frac{c_{n-1}}{c_n} & \frac{c_{n-1}}{c_n} & \dots & \dots & \frac{c_0}{c_n} \\ -1 & x - x_{n-2} & 0 & \dots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & x - x_0 \end{pmatrix}$$

We expand the determinant around the terms at $(1, 1)$ and $(2, 1)$.

$$f_n = \left((x - x_{n-1}) + \frac{c_{n-1}}{c_n}\right) det\begin{pmatrix} x - x_{n-2} & 0 & \dots & \dots & 0 \\ -1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & x - x_0 \end{pmatrix}$$

$$-(-1)\, det\begin{pmatrix} 0 + \frac{c_{n-2}}{c_n} & \dots & \dots & \dots & \frac{c_0}{c_n} \\ -1 & x - x_{n-3} & 0 & \dots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & x - x_0 \end{pmatrix}$$

The determinant of the matrix on the left is simply $\prod_{i=0}^{n-2}(x - x_i)$. The matrix on the right is, by assumption, simply $f_{n-1}(x)$ except with the $(x - x_{n-1})$ term equal to 0, and an c_n element where the c_{n-1} element usually is. Thus,

$$f_n = ((x - x_{n-1}) + \frac{c_{n-1}}{c_n})\prod_{i=0}^{n-2}(x - x_i) + \left(\frac{c_0}{c_n} + \frac{c_1}{c_n}(x - x_0) + \dots + \frac{c_{n-2}}{c_n}(x - x_0)\dots(x - x_{n-3})\right)$$

$$= (x - x_{n-1})\dots(x - x_0) + \frac{c_{n-1}}{c_n}(x - x_{n-2})\dots(x - x_0) + \dots + \frac{c_1}{c_n}(x - x_0) + \frac{c_0}{c_n} = \frac{1}{c_n}P_n(x),$$

which is what we wanted to show. □

Thus, M_n as defined above is the companion matrix for the Newton form polynomial P_n. As an extra check on our solution, we note that when $x_0 = \dots = x_{n-1} = 0$, then M_n reduces to the companion matrix for $c_0 + c_1x + \dots + c_nx^n$ given in the problem set statement.

Problem 3 *(Source code for this problem attached as* problem3.py, problem3.m, *and* matlab-eigenvalues.txt*)*

We define some terms. Let

$$M_n = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_n \end{pmatrix},$$

where $a_i = 0$, $b_i^2 = \frac{i^2}{(2i-1)(2i+1)}$ and $f_n(x) = det(xI_n - M_n)$. We will prove that $f_n(x) = \alpha_n P_n(x)$, where

$$\alpha_n = \prod_{i=1}^{n} \frac{n}{2n-1},$$

and $P_n(x)$ is the n^{th} Legendre polynomial, which can be defined recursively by

$$P_n(x) = x\frac{2n-1}{n}P_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x),$$

with $P_0 = 1$ and $P_1 = x$.

Conjecture.

$$f_n(x) = \alpha_n P_n(x).$$

Proof. By induction. Our anchors are at $n = 1$ and $n = 2$.

$$f_1(x) = x = \alpha_1 P_1(x)$$

$$f_2(x) = det\begin{pmatrix} x & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & x \end{pmatrix} = x^2 - \frac{1}{3} = \frac{2}{3}P_2(x) = \alpha_2 P_2(x)$$

Now we must show that $f_{n-2}(x) = \alpha_{n-2}P_{n-2}(x)$ and $f_{n-1}(x) = \alpha_{n-1}P_{n-1}(x) \Rightarrow f_n(x) = \alpha_n P_n(x)$. Consider the relation between the determinants of a symmetric tridiagonal matrix given in the statement of the problem. We know that

$$det(xI_n - M_n) = (x - a_n)det(xI_{n-1} - M_{n-1}) - (-b_{n-1})^2 det(xI_{n-2} - M_{n-2}).$$

If we set $a_n = 0$, and let $b_{n-1}^2 = \frac{(n-1)^2}{(2n-1)(2n-3)}$, we have

$$f_n(x) = xf_{n-1}(x) - \frac{(n-2)^2}{(2n-1)(2n-3)}f_{n-2}(x)$$

$$f_n(x) = x\alpha_{n-1}P_{n-1}(x) - \frac{(n-1)^2}{(2n-1)(2n-3)}\alpha_{n-2}P_{n-2}(x)$$

We extract an $\alpha_n = \frac{n}{2n-1}\alpha_{n-1} = \frac{n}{2n-1}\frac{n-1}{2n-3}\alpha_{n-2}$ term:

$$f_n(x) = \alpha_n\left(x\frac{2n-1}{n}P_{n-1}(x) - \frac{(n-1)^2}{(2n-1)(2n-3)}\frac{(2n-1)}{n}\frac{(2n-3)}{(n-1)}P_{n-2}(x)\right)$$

$$f_n(x) = \alpha_n \left(x \frac{2n-1}{n} P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x) \right),$$

and by the recursive definition of the Legendre polynomial,

$$f_n(x) = \alpha_n P_n(x),$$

which is what we wanted to show. $\square$

Thus, the eigenvalues of M_{32} (with $a_i = 0$ and $b_i^2 = \frac{i^2}{(2i-1)(2i+1)}$) are the roots of P_{32}. After numerically calculating and sorting the eigenvalues, we get the following list:

-0.99726386	-0.98561151	-0.96476226	-0.93490608
-0.89632116	-0.84936761	-0.7944838	-0.73218212
-0.66304427	-0.58771576	-0.50689991	-0.42135128
-0.3318686	-0.23928736	-0.14447196	-0.04830767
0.04830767	0.14447196	0.23928736	0.3318686
0.42135128	0.50689991	0.58771576	0.66304427
0.73218212	0.7944838	0.84936761	0.89632116
0.93490608	0.96476226	0.98561151	0.99726386

These eigenvalues are calculated using SciPy's *linalg.eig()* function. As given, $\sim .239287$ is in fact one of the zeros. (For comparison's sake, I've attached a printout of MATLAB's output of the eigenvalues.)

Problem 4 *(Source code attached as* problem4.py*)*

Let $p_N(x) = c_0 + \ldots + c_N x^N$ be the degree N minimax polynomial of $sin(x)$ on $[0, \frac{\pi}{2}]$, and let

$$\epsilon_N = max_{0 \le x \le \frac{\pi}{2}} \mid sin(x) - p_N(x) \mid .$$

We want to find $p_7(x)$. We first note that by choosing seven points $\{x_i\}$ (with $x_0 = 0$ and $x_8 = \frac{\pi}{2}$) such that

$$sin(x_i) - p_7(x_i) = (-1)^i h, \quad 0 \le i \le 8,$$

we can write nine equations in $(c_0, \ldots, c_8, h)$:

$$c_0 + c_1 x_0 + \ldots c_7 x_0^7 + h = sin(x_0)$$

$$c_0 + c_1 x_1 + \ldots c_7 x_1^7 - h = sin(x_1)$$

$$\vdots$$

$$c_0 + c_1 x_8 + \ldots c_7 x_8^7 + h = sin(x_8)$$

These correspond to the matrix equation

$$\begin{pmatrix} 1 & \ldots & x_0^7 & 1 \\ 1 & \ldots & x_1^7 & -1 \\ \vdots & & \vdots & \vdots \\ 1 & \ldots & x_0^7 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_7 \\ h \end{pmatrix} = \begin{pmatrix} sin(x_0) \\ sin(x_1) \\ \vdots \\ sin(x_8) \end{pmatrix}$$

We can solve this using software. Thus, given a set of seven points, we can obtain a seventh-degree polynomial that misses $sin(x)$ by h at each of the points. We intend to find the first 8 digits of the $\{c_i\}$. We make an initial guess $\{x_i\}$, and then refine our guess (a better guess will result in a lower ϵ_7) until we have the required number of digits. The method of refining out initial guess is as follows:

Given a set $\{x_i\}$ (and an h and $\{c_i\}$ derived from it), calculate ϵ_N, and also the point x' at which $sin(x') - (c_0 + \ldots + c_7 {x'}^7) = \epsilon_N$. Since the calculated value of h is typically smaller than ϵ_N, if we move one of the x_i to x', we will eliminate this source of error. We decide to move the x_i closest to x'.

With this method in mind, and the initial guess $\{x_i\} = .2, .4, .6, .8, 1, 1.2, 1.4$, we follow these steps:

Step 1. Given $\{x_i\}$, calculate $\{c_i\}$ to construct a polynomial $p_7(x)$.

Step 2. Calculate ϵ_7, the maximum error between $sin(x)$ and $p_7(x)$. Note where the error is happening as x'.

Step 3. Move the closest x_i to x'.

Step 4. Go back to Step 1, and repeat until the $\{c_i\}$ converge.

In less than twenty iterations, the process settles on (to as many digits as given by SciPy):

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.0612045080772363\,47 \\ 0.23474608626153651 \\ 0.49251533428490585 \\ 0.79392758745194458 \\ 1.092861124015301 \\ 1.344620505292353 \\ 1.5121082344331855 \\ \frac{\pi}{2} \end{pmatrix}, \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ h \end{pmatrix} = \begin{pmatrix} -.0000000195367731583 \\ 1.00000155323 \\ -.0000202292837008 \\ -.166566787675 \\ -.000239703970351 \\ .00863920572651 \\ -.000205700049894 \\ -.000137323212602 \\ .0000000195367731583 \end{pmatrix}$$

Our calculated ϵ_7 is .000000019536773532280449, which agrees with h to eight significant digits. This is reassuring, because we know that by the definition of the minimax function, $h = \epsilon_7$. To eight digits, our minimax function of degree 7 of $sin(x)$ on $[0, \frac{\pi}{2}]$ is

$$p_7(x) = (-1.9536773 \cdot 10^{-8}) + (1.0000015)x + (-2.0229283 \cdot 10^{-5})x^2$$

$$+(-.16656678)x^3 + (-2.3970397 \cdot 10^{-4})x^4 + (8.6392057 \cdot 10^{-3})x^5$$

$$+(-2.0570005 \cdot 10^{-4})x^6 + (-1.3732321 \cdot 10^{-4})x^7.$$

(As an afterthought, this is somewhat close to the seventh Taylor polynomial of $sin(x)$, which is, to eight digits, $x - .16666666x^3 + .0083333333x^5 - 0.00019841269x^7$. Well, more in the first terms than in the latter ones.)